# ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

R. P. BOAS, JR.

1. **Introduction.** An entire function $f(z)$ is said to be of exponential type if it is of order one and mean type; that is, if for some non-negative $c$ and every positive $\epsilon$ there is a number $A(\epsilon)$ such that

$$|f(z)| < A(\epsilon)e^{(c+\epsilon)|z|} \tag{1}$$

for all $z$. The smallest $c$ which can be used in (1) is called the type of $f(z)$. An alternative definition[1] is

$$c = \limsup_{n\to\infty} |f^{(n)}(z)|^{1/n}; \tag{2}$$

it is immaterial which value of $z$ is used in (2). If (1) holds in a region of the $z$-plane, for example in an angle, $f(z)$ is said to be of exponential type $c$ in that region.

Functions of exponential type have been extensively studied, both for their own sake and for their applications. I shall discuss here a selection of their properties, chosen to illustrate how the restriction (1) on the growth of a function restricts its behavior in other ways.[2]

2. **Representations.** Various formulas are available for representing functions of exponential type. Some of these representations are useful for deriving results of the kind discussed later in this report; and they are of considerable interest for their own sake.

If $f(z)$ is an entire function satisfying (1), there is a function $\phi(w)$, analytic in $|w| > c$, such that

$$f(z) = \int_C e^{zw}\phi(w)dw, \tag{3}$$

where $C$ is any contour containing $|w| = c$ in its interior.[3] The function $\phi(w)$ is defined by either of the equivalent formulas

$$\phi(w) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{n!a_n}{w^{n+1}} \quad \text{if} \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

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An address delivered before the New York meeting of the Society on April 4, 1942, by invitation of the Program Committee; received by the editors July 28, 1942.

[1] See [10, p. 241].

[2] For a report on functions of exponential type from another point of view, see [10].

[3] See [25, pp. 578–586].

$$\phi(w) = \frac{1}{2\pi i}\int_0^\infty f(x)e^{-xw}dx, \qquad \Re(w) > c.$$

More precisely, $C$ in (3) can be any curve surrounding the set of singular points of $\phi(w)$; the configuration of this set depends on the growth of $f(z)$ in various directions. In this report, however, I shall usually neglect the more refined results in which the shape of the set of singular points of $\phi(w)$ is taken into consideration. Any result involving the type of $f(z)$ will have refinements involving the growth of $f(z)$ in two or more directions.

However, the case in which $\phi(w)$ has singularities only along a line segment is of particular interest. If $f(z)$ belongs to $L^2$ on a line (for definiteness, the real axis), then

$$f(z) = \int_{-c}^{c} e^{izt}F(t)dt, \qquad F(t) \in L^2(-c, c); \tag{4}$$

and conversely.[4] This result can be extended to other $L$-classes, though with some loss of elegance. If $f(z)\in L^p(-\infty, \infty)$ $(1<p<2)$, then $f(z)$ has the representation (4) with $F(t)\in L^{p/(p-1)}$, but the converse is false.[5] A necessary and sufficient condition [**23**, p. 111] for $f(z)$ to belong to $L^p$ $(p>1)$ is that

$$f(z) = z\int_{-c}^{c} [F(c) - F(t)]e^{izt}dt - 2F(c)z^{-1}\sin cz$$

where $F(t)$ is a continuous function of period $2c$ whose Fourier coefficients $c_n$ satisfy

$$\sum_{-\infty}^{\infty} |nc_n|^p < \infty.$$

When $p=1$, we have the necessary and sufficient condition [**1**, p. 283]

$$f(z) = \int_{-c}^{c} G(t)e^{izt}dt,$$

where $G(t)$ has an absolutely convergent Fourier series and $G(c-) = G(-c+) = 0$.

For functions $f(z)$ bounded on the real axis, we have the representations (necessary but not sufficient)[6]

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[4] [**21**, p. 13]. A sharper result is given by Plancherel and Pólya [**23**, p. 228].

[5] [**1**, p. 280]. The result is implicit in [**23**, p. 243].

[6] [**3**, p. 151]. To obtain (5), apply (4) to $z^{-1}[f(z)-f(0)]$.

$$f(z) = f(0) + z\int_{-c}^{c} e^{izt}F(t)dt, \qquad F(t) \in L^2; \tag{5}$$

and also [2]

$$f(z) = \int_{-c}^{c} e^{izt}\frac{d^2F(t)}{dt}; \tag{6}$$

the notation in (6) is purely formal; what is meant is that the 2-transform $F(t)$ (in the sense of Bochner [9]) of $f(z)$ is necessarily linear outside $(-c, c)$. Both (5) and (6) have been applied in the study of functions of exponential type [19, 3].

For a function of exponential type in an angle, Macintyre [20] has given a useful representation by a contour integral.

3. **Operators.** A large number of the results mentioned in this report have to do with relations between the behavior of an entire function $f(z)$ and that of the functions obtained by operating on $f(z)$ with a sequence of operators $L_n$. In many cases, the $L_n$ are the powers $L^n$ of a single operator. The most interesting—certainly the most easily dealt with—operators $L$ appear to be those which are distributive and (in a suitable sense) continuous, which transform an entire function of exponential type less than $c$ into another function of type less than $c$, and which are permutable with differentiation. Associated with any such operator $L$ there is a function $\lambda(w)$, analytic in $|w| < c$, such that if $f(z)$ has the representation (3), then[7]

$$L[f(z)] = \int_C e^{zw}\phi(w)\lambda(w)dw; \tag{7}$$

correspondingly, $L$ may be regarded as the differential operator of infinite order, $\lambda(d/dz)$. Examples which have been frequently discussed are $L[f(z)]=f(z+1)$, $L[f(z)]=f'(z)$, $L[f(z)]=f(z+1)-f(z)$. The corresponding functions $\lambda(w)$ are $e^w$, $w$, and $e^w-1$.

Corresponding to any sequence of operators $L_n$ of this character, there is a uniqueness problem: given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$, for what class of functions of exponential type does $L_n[f(a_n)]=0$ $(n=0, 1, 2, \cdots)$ imply $f(z)\equiv 0$? A somewhat more general problem asks, given a set of circles $C_n$, whether at least one of the functions $L_n[f(z)]$ is necessarily univalent in the corresponding $C_n$.[8]

Because of the representation (7) for the operators $L_n$, uniqueness

[7] Boas (unpublished).

[8] This is really a generalization of the $L_n$ problem for $f'(z)$.

theorems reduce to theorems of closure for sets of analytic functions. If every function analytic inside and on the contour $C$ can be expanded in a uniformly convergent series of the functions $e^{a_n w}\lambda_n(w)$, then in particular the function $e^{zw}$ can be so expanded,

$$e^{zw} = \sum_{n=0}^{\infty} c_n(z)e^{a_n w}\lambda_n(w).$$

Substituting this expansion into (3) and integrating term by term, we obtain, if $L_n[f(a_n)]=0$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z) \int_C e^{a_n w}\lambda_n(w)\phi(w)dw = \sum_{n=0}^{\infty} c_n(z)L_n[f(a_n)] = 0.$$

Similarly, the generalized uniqueness problem involving univalence leads to the closure problem for the set $\lambda_n(w)(e^{a_n w}-e^{b_n w})$, where $a_n$ and $b_n$ are arbitrary (unequal) points in the circles $C_n$.

4. **Uniqueness theorems.** The most elementary uniqueness theorem is that associated with $L_n=L^n$, $L[f(z)]=f'(z)$, $\{a_n\}=\{0\}$. Here the theorem states that a function of exponential type—or, of course, any analytic function—which vanishes with all its derivatives at a given point, vanishes identically. An immediate generalization is Whittaker's problem: if $f^{(n)}(a_n)=0$, $|a_n|\leqq 1$, what is the largest number $A$ such that if $f(z)$ is of type less than $A$ it must vanish identically [32, p. 45]? Since the functions $z^n e^{a_n z}$ are known [5, pp. 481–482; 15] to form a closed set in $|z|<\log 2$, the general argument just outlined shows that[9] $A\geqq\log 2$. Pólya has recently shown[10] that $\log 2 < A < 0.77 < \pi/4 = 0.785+$.[11] If the $a_n$ are further restricted to be real, the corresponding constant $A$ is actually $\pi/4$ [28]. If we make the restriction $a_{2n+1}=0$, $|a_{2n}|\leqq 1$, then[12] the corresponding $A$ exceeds 1.31.

The univalence theorem corresponding to Whittaker's problem is that one at least of the derivatives of $f(z)$ must be univalent in the unit circle (an infinite number, if $f(z)$ is not a polynomial), provided that $f(z)$ is of type less than $A$, where $A\geqq\log 2$ [6, 18]; the best possible $A$ is not known.

If $L_n[f(z)]=f^{(n)}(z)+b_n f(0)$, where the $b_n$ are sufficiently small num-

[9] First proved in another way by Takenaka [29].

[10] Unpublished result.

[11] Thus disproving the conjecture that $A=\pi/4$. The example showing that $A<\pi/4$ is $f(\pi z/4)$, where $f(z)=e^z+\omega^{-1}e^{\omega z}+e^{\omega^2 z}$ and $\omega^3+\omega+1=0$.

[12] The result $A\geqq 0.78$ is given in [5, p. 486]; the result stated here can be obtained by modifying the argument on p. 482.

bers, we obtain a theorem [5, p. 485] stating that the numbers $f^{(n)}(a_n)$ cannot all be very small unless $f(z)\equiv 0$. If $L_{2n}[f(z)]=L_{2n+1}[f(z)]=f^{(2n)}(z)$, $a_{2n}=0$, $a_{2n+1}=1$, we have the uniqueness theorem for the so-called two-point boundary problem: if every derivative of even order of a function of exponential type less than or equal to $c$ vanishes both at 0 and at 1, then $f(z)$ is a sine polynomial of order at most $c/\pi$; if $c<\pi$, $f(z)\equiv 0$.[13] A similar uniqueness theorem holds for more general sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$, near 0 and 1, respectively; the corresponding univalence result also is true.

5. **Further uniqueness theorems.** When the function $\lambda(w)$ of (7) is univalent inside the contour $C$, we can establish the closure of the functions $\lambda^n(w)$ by making a conformal map which carries them into the functions $z^n$ [14]. In this way it is found, for example, that the operators $L_n[f(a_n)]=\Delta^n f(0)$ have a uniqueness theorem for functions of type less than log 2; from this result it can be shown that a function of type less than log 2, taking integral values at the points $n=0, 1, 2, \cdots$, is necessarily a polynomial.[14] More generally, the operators $L_n[f(a_n)]=\Delta^n f(a_n)$ can be studied; for example, the critical type for the operators $\Delta^n f(a_n)$, $|a_n|\leqq 1$, is at least log 3/2.

If $\lambda(w)$ is not only univalent, but also has no zeros inside the contour $C$, the sequence of operators $L^n$ can be imbedded in the family of operators $L^t$ corresponding to the functions $[\lambda(w)]^t$, and uniqueness theorems obtained corresponding to various sequences $L^{\lambda_n}$. A great deal of attention has been paid to the operator $L[f(z)]=f(z+1)$, corresponding to the function $\lambda(w)=e^w$; results for this operator naturally can be transformed by conformal mapping so as to apply to other operators. With this operator the uniqueness problem can be formulated more simply as the problem of finding conditions under which $f(\lambda_n)=0$ implies $f(z)\equiv 0$.

The oldest result of this kind is Carlson's theorem: If $f(z)$ is of exponential type $c$, $c<\pi$,[15] then $f(n)=0$ $(n=0, 1, 2, \cdots)$ implies that $f(z)\equiv 0$.[16] A more delicate result, applying only to entire functions, states that if

$$(8)\quad |f(z)|<\epsilon(|z|)e^{\pi|z|}(1+|y|)^{-2\delta},\quad 0<\delta<1/2,\quad \lim_{r\to\infty}\epsilon(r)=0,$$

[13] [27]. Another proof in [7]. For generalizations, see [14].

[14] Pólya and Hardy; see [32, p. 55].

[15] Or, more generally, of (some) exponential type in a right half-plane and of type less than $\pi$ along its boundary.

[16] See, for example, [30, p. 186]. A related result of Estermann (Dienes [13, p. 259]) is that if $f(z)$ is of type less than 1, and $f(z_n)=0$, $|z_n|=n$ $(n=0, 1, 2, \cdots)$, then $f(z)\equiv 0$.

and $f(\lambda_n)=0$ $(n=0, 1, 2, \cdots)$, where $|\lambda_n-n|<\delta$, then $f(z)\equiv 0$.[17] More general sequences $\{\lambda_n\}$ are allowable if we go back to the stronger requirement that $f(z)$ is of type $c$, $c<\pi$. For example, if $f(z)$ is of type less than $\pi$ in the half-plane $x\geqq 0$, and if $f(\lambda_n)=0$ where the real numbers $\lambda_n$ satisfy $\lim_{n\to\infty} n/\lambda_n=D\geqq 1$, and $|\lambda_n-\lambda_m|\geqq|n-m|\delta$, $\delta>0$, then $f(z)\equiv 0$.[18] Much more general sequences $\{\lambda_n\}$ can be used if the growth of $f(z)$ is restricted by a condition stating roughly that $f(z)$ is of order less than one along the real axis. More precisely,[19] if $f(z)$ is an entire function of exponential type $c$, such that

$$\lim_{R\to\infty}\int_1^R (1/x^2)\log|f(x)f(-x)|\,dx$$

exists and is finite, and

$$\limsup_{x\to\infty}(1/x)\log|f(\pm x)|\leqq 0,$$

then the zeros of $f(z)$ in $x\geqq 0$ and in $x\leqq 0$ have a density $B$ at most equal to $c/\pi$. Consequently we cannot have $f(\lambda_n)=0$ for a sequence $\{\lambda_n\}$ whose density exceeds $c/\pi$.

Even more generally, the hypothesis that $f(\lambda_n)=0$ can be replaced by one stating that the numbers $f(\lambda_n)$ are very small;[20] for example, that $f(\lambda_n)=O(e^{-\delta\lambda_n})$, $\delta>0$.

Another interesting result is that if (8) is satisfied with $\delta=1$, then $f(z)\equiv 0$ if $f(z)$ vanishes at least once in every interval $(n, n+1)$ [31, p. 213]. This suggests that if we let the zeros coalesce in pairs, so that $f(z)$ and $f'(z)$ both vanish at $z=2n$, then $f(z)\equiv 0$ if $f(z)$ is of type less than $\pi$. More generally, we can consider the case where we have two operators $L$ and $M$, $L[f(2n)]=M[f(2n)]=0$, and the functions $\lambda$ and $\mu$ associated with $L$ and $M$ satisfy suitable conditions [4].

6. **Growth theorems.** Let $L_t$ be a one-parameter family of operators, for example the powers of an operator $L$ whose $\lambda(w)$ has no zeros. A uniqueness theorem states that, given sequences $\{a_n\}$ and $\{\lambda_n\}$, any function $f(z)$ of sufficiently small type, such that $L_{\lambda_n}[f(a_n)]=0$, vanishes identically. If $f(z)$ is identically zero, then of course so are $L_t[f(z)]$ and $M_t[f(z)]$ for every $t$, where $M_t$ is another family of operators. We now suppose, not that the elements of the sequence $\{L_{\lambda_n}[f(a_n)]\}$ are zero, but their growth is restricted in some way;

[17] [3, p. 158]. The case $\delta=0$ is due to Pólya [24] and Valiron [31, p. 204].

[18] This is an easy deduction from Carleman's theorem (for Carleman's theorem see, for example [30, p. 130]).

[19] See [17, especially pp. 13, 25].

[20] Levinson [17, p. 19]; Levinson gives a number of much more general results.

and we ask what can be said about the growth of a sequence $\{M_{\mu_n}[f(b_n)]\}$. The case most extensively studied has been that in which $a_n=b_n=0$; in this case, it is more convenient to think of the problem as that of determining the growth properties of $\{M^*[f(\mu_n)]\}$ from those of $\{L^*[f(\lambda_n)]\}$, where $L^*[f(z)]=L_z[f(0)]$ and $M^*$ is defined similarly.

It is not always desirable to restrict $\{\lambda_n\}$ and $\{\mu_n\}$ to be countable sequences. If we replace both sequences by the set of all real numbers $x$, our original uniqueness problem becomes trivial; but we have the following non-trivial growth problem: given that $L[f(x)]$ is bounded for real $x$, what bound can be assigned to $M[f(x)]$, where $L$ and $M$ are given operators? The oldest case of the problem is S. Bernstein's problem of the bound for the derivative of a function, given a bound for the function; here $L$ is the identity operator and $M[f(z)]=f'(z)$. Here we have the result that $|f'(x)| \leqq cK$ if $|f(x)| \leqq K(-\infty<x<\infty)$.[21] More generally, if $M$ is an operator with associated function $\mu(w)$ analytic in $|w|<c$, then for every $f(z)$ of type $c'<c$ there is a number $A(c')$, independent of $f$, such that[22]

$$|M[f(x)]| \leqq A(c') \sup_{-\infty<x<\infty} |f(x)|. \tag{9}$$

Under suitable restrictions on $\mu(w)$ on the imaginary axis, it can be shown by use of the representation (5) that $A(c')$ may be replaced by a number $A(c)$ depending only on $c$. Still more generally we can obtain results of the form

$$|M[f(x)]| \leqq A(c') \sup_{-\infty<x<\infty} |L[f(x)]|.$$

Such a result is no more general than (9) if $L$ has an inverse, that is, if $\lambda(w)$ has no zeros; but may hold even when $\lambda(w)$ does have zeros if the zeros are off the imaginary axis.

7. **Growth theorems involving sequences.** The theorems to be discussed in this section deduce the growth properties of $M[f(x)]$ from those of $L[f(\lambda_n)]$. The necessary restrictions on the sequence $\{\lambda_n\}$ vary with the expression taken as measure of the rate of growth. We consider first functions bounded on the real axis. If $f(z)$ is of type $c$, $c<\pi$, and $|f(n)| \leqq K$ $(n=0, \pm 1, \cdots)$, then[23] $|f(x)| \leqq A(c)K$; the best possible $A(c)$ is[24] $O(-1/\log(\pi-c))$ as $c\to\pi$. This result (apart from

[21] See [26, vol. 2, p. 35, section IV, problem 201].

[22] Civin [12], for a restricted class of functions of exponential type. The results can be extended to the general case by the method used in [3, pp. 150–152].

[23] Cartwright [11]. Other proofs by Pfluger [22], Macintyre [20], Boas [3].

[24] Boas and Schaeffer [8].

the evaluation of $A(k)$) extends to functions of exponential type in a half-plane,[25] and in that form carries over directly to other rates of growth, for example, $f(x)=O(x^p)$ $(p>0)$ or $f(x)=O(e^{Ax})$. However, results involving an exponential rate of growth for $f(x)$ on the real axis extend to more general sequences of points than do those involving merely boundedness for $f(x)$. For boundedness, the most general result so far obtained is that if $|\lambda_n-n|<A$ and $|\lambda_m-\lambda_n| \geqq \delta(m\neq n;\ m,\ n=0,1,2,\cdots)$, then $f(z)$, of type less than $\pi$ in $x>0$ and bounded at $\{\lambda_n\}$, is bounded on the real axis.[26] Results of Levinson [17, p. 127] show that one cannot go much farther in this direction. On the other hand, results of the type

$$\limsup_{x\to\infty}\frac{\log|f(x)|}{x}=\limsup_{n\to\infty}\frac{\log|f(\lambda_n)|}{\lambda_n}$$

hold for sequences $\{\lambda_n\}$ satisfying such general conditions as[27]

$$|\lambda_n-\lambda_m|\geqq|n-m|\,d, \qquad d>0,\ n/\lambda_n\to D,$$

where the type of $f(z)$ is less than $\pi D$.

In these results quoted so far in this section, the operators $L$ and $M$ have both been the identity operator. It is clear that more general theorems, at least for the case of boundedness, can be deduced indirectly from the results of §6. For, from the boundedness of $f(\lambda_n)$ we can infer first that of $f(x)$ and then that of $L[f(x)]$. In this way we can show, for example, that if $|f(n)|\leqq K$ and $f$ is of type $c'<\pi$, then $|f'(x)|\leqq c'A(c')K$, where $A(c')$ is the same as in (9). However, we may in some cases obtain sharper results by proceeding directly; thus Macintyre has pointed out [20, p. 6] that $|f(n)|\leqq K$ implies

$$|f''(x)+\pi^2 f(x)|\leqq AK, \tag{10}$$

where $A$ is a constant, even when $f(z)$ is of type $\pi$. The reason for the difference between the operators $L_1[f]=f'$ and $L_2[f]=f''+\pi^2 f$ is clearly shown by the difference between the corresponding functions $\lambda_1(w)=w$ and $\lambda_2(w)=w^2+\pi^2$. The second vanishes at $w=\pm i\pi$; it is generally true that an $L$ whose $\lambda$ has this property has the stronger boundedness property typified by (10).

[25] Cartwright [11], Macintyre [20].

[26] Duffin and Schaeffer (unpublished). I am indebted to these authors for permission to include their result in this report.

For a theorem in which the boundedness of $f(z)$ on two sequences implies its boundedness on two lines (and hence that it is a constant) see Levinson [17, p. 122].

[27] For this, and much more general results, see Levinson [17, pp. 100 ff.]. Other theorems of this character are given by Junnila [16].

8. **Integrability.** Bernstein's theorem on derivatives states in effect that a function of exponential type, if bounded on a line, cannot oscillate too rapidly on the line. Another illustration of this fact is that if $f(z)$ is of type less than $\pi$ and is bounded on the average near the integers, it is bounded on the real axis; more precisely, if

$$\sup_{-\infty<a<\infty} \int_{a-\epsilon}^{a+\epsilon} \psi(|f(t)|)dt < \infty,$$

where $\epsilon \geqq 0$, $\delta \geqq 0$, $\epsilon+\delta>0$, and $\psi(t)$ is non-decreasing and unbounded, then $f(x)$ is bounded [6, p. 163]. In particular, $f(x)$ is bounded (and hence approaches zero) on the real axis if it belongs to $L^p$ for some positive $p$ [23, p. 124].

A result of similar character is that

$$\sum_{-\infty}^{\infty} |f(x_n)|^p < \infty, \qquad p > 0, \tag{11}$$

with

$$|x_n - n| \leqq L < (2\pi^2)^{-1}$$

implies

$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty \tag{12}$$

if[28] $f(z)$ is of type less than $\pi$; conversely, if $f(z)$ is of any finite type and if the number of $x_n$'s in any interval of unit length is bounded, then (12) implies (11).[29]

Still another indication of the restrictions imposed on the oscillation of $f(x)$ by its integrability properties is that if $f(x)$ belongs to $L^p$ $(p>0)$, so does $f'(x)$ [23, p. 127]. This can be considered as a generalization of Bernstein's theorem mentioned in §6; Bernstein's theorem is the limiting case $p\to\infty$.

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DUKE UNIVERSITY

# SOME NOTES ON AN EXPANSION THEOREM OF PALEY AND WIENER

R. J. DUFFIN AND J. J. EACHUS

Paley and Wiener[1] have formulated a criterion for a set of functions $\{g_n\}$ to be "near" a given orthonormal set $\{f_n\}$. The interest of this criterion is that it guarantees the set $\{g_n\}$ to have expansion properties similar to an orthonormal set.[2] In particular, they show that the set $\{g_n\}$ approximately satisfies Parseval's formula. In the first part of this paper we show that, conversely, if a set $\{g_n\}$ approximately satisfies Parseval's formula then there exists at least one orthonormal set which it is "near."

In the second part of the paper we consider sets which are on the borderline of being near a given orthonormal set.

The last part of this paper gives a simple formula for constructing sets near a given orthonormal set. As an application of this formula we obtain new properties of the so called non-harmonic Fourier series.

We shall handle these problems abstractly, using the notation of Hilbert space.[3] Subscript variables are assumed to range over all positive integers and $\sum$ shall mean a sum over all positive integers. By a finite sequence shall be meant a sequence with only a finite number of nonzero members. For application to the space $L_2$ the norm of a function $f(x)$ is defined in the usual way as $\|f\| = (\int_a^b |f(x)|^2 dx)^{1/2}$. A complete set which satisfies the Paley-Wiener criterion shall be termed strongly complete.

The principal novelty in the proof is the association of a linear transformation $G$ with each set of elements $\{g_n\}$. Thus if $\{\psi_n\}$ is an orthonormal set we define $G\sum a_n\psi_n = \sum a_n g_n$ for every finite sequence of constants $\{a_n\}$. The norm of $G$ is the limit superior of $\|Gx\|$ for elements $x$ such that $\|x\| = 1$. With this definition of norm the aggregate of bounded linear transformations clearly forms a normed linear

Presented to the Society, April 12, 1940 under the title *Converse of a closure theorem of Paley and Wiener*, and September 5, 1941 under the title *A Paley-Wiener type expansion theorem*; received by the editors October 8, 1941.

[1] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 100.

[2] R. P. Boas, Jr., Journal of the London Mathematical Society, vol. 14 (1939), p. 242; Duke Mathematical Journal, vol. 6 (1940), p. 148; American Journal of Mathematics, vol. 63 (1941), p. 361.

[3] Because of the difficulty of finding adequate references to non-separable Hilbert space we confine ourselves to separable space. However, our theorems remain true for non-separable Hilbert space provided the range of subscript variable is redefined.

space. It is in this space that the concept of one sequence of functions being "near" another is made precise; thus two points are "near" if they are within unit distance of each other.

1. **A converse theorem.** We prove the following theorem.

THEOREM A. *Suppose that the set of elements* $\{g_n\}$ *satisfies the relation*

$$(1) \qquad (1-\theta)(\sum |a_n|^2)^{1/2} \leqq \|\sum a_n g_n\| \leqq (1+\theta)(\sum |a_n|^2)^{1/2}$$

*for every finite sequence of constants* $\{a_n\}$ *and for some fixed number* $\theta$; $0 \leqq \theta < 1$. *Then there exists an orthonormal set* $\{f_n\}$ *such that*

$$(2) \qquad \|\sum a_n(f_n - g_n)\| \leqq \theta(\sum |a_n|^2)^{1/2}.$$

PROOF. Relation (1) guarantees that the set $\{g_n\}$ is linearly independent; hence the closure of all linear combinations of the set $\{g_n\}$ is an infinite-dimensional manifold, a Hilbert space. Let $\{\psi_n\}$ be some complete orthonormal set of this Hilbert space and define the transformation $G$ by $G\sum a_n\psi_n = \sum a_n g_n$. Thus if $y = \sum a_n\psi_n$ we may express relation (1) in the form

$$(3) \qquad (1-\theta)\|y\| \leqq \|Gy\| \leqq (1+\theta)\|y\|.$$

The following lemmas are well known:[4]

LEMMA 1. *A bounded linear transformation whose range and domain both determine Hilbert space may be factored in the form* $FP$ *where* $F$ *is a unitary transformation and* $P$ *is a positive definite self-adjoint transformation.*

LEMMA 2. *If* $S$ *is a self-adjoint transformation then*

$$\|Sx\|/\|x\|$$

*and*

$$|(Sx, x)|/(x, x)$$

*have the same upper and lower bounds.*

Clearly $G$ satisfies the conditions of Lemma 1, so $G = FP$. From relation (3) it follows that

$$(1-\theta)\|y\| \leqq \|Py\| \leqq (1+\theta)\|y\|.$$

[4] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, 1932; J. von Neumann, Annals of Mathematics, (2), vol. 33 (1932), p. 308; A. Wintner, Mathematische Annalen, vol. 37 (1933), p. 257.

Moreover, this inequality clearly remains valid if $y$ is an arbitrary element; so by Lemma 2

$$(1-\theta)(y, y) \leqq (Py, y) \leqq (1+\theta)(y, y).$$

This may be written as

$$-\theta(y, y) \leqq (Py-y, y) \leqq \theta(y, y).$$

Again using Lemma 2, we have $\|y-Py\| \leqq \theta\|y\|$. Let us define $f_n = F\psi_n$. Because $F$ is unitary it follows that $\{f_n\}$ is an orthonormal set. Moreover,

$$\begin{aligned}\|\sum a_n(f_n-g_n)\| &= \|Fy-Gy\| \\ &= \|F(y-Py)\| \\ &= \|y-Py\| \\ &\leqq \theta\|y\| = \theta(\sum|a_n|^2)^{1/2}.\end{aligned}$$

2. **The borderline case, $\theta=1$.** We now establish this theorem.

THEOREM B. *The set of elements $\{g_n\}$ and the orthonormal set $\{f_n\}$ satisfy*

$$\|\sum a_n(f_n-g_n)\| < (\sum|a_n|^2)^{1/2} \tag{4}$$

*for every sequence of coefficients $\{a_n\}$ such that*

$$0 < \sum|a_n|^2 < \infty.$$

*Then $\{g_n\}$ is complete if $\{f_n\}$ is complete.*[5]

PROOF. If $\{g_n\}$ is not complete, there exists an element $z$ such that $(g_n, z)=0$ for all $n$. We can express $z$ in the form $z=\sum a_n f_n$, where $0<\sum|a_n|^2<\infty$. Thus

$$\|\sum a_n(f_n-g_n)\|^2 = \|z\|^2 - 2\Re\sum a_n(g_n, z) + \|\sum a_n g_n\|^2 \geqq \|z\|^2.$$

This contradicts (4), and the theorem is proved.

The following counterexample shows that the completeness of the set $\{g_n\}$ does not imply that the orthonormal set $\{f_n\}$ is complete. Let $\{c_n\}$ be a sequence of constants such that $c_n > c_{n+1} > 1$. Let $\{\psi_n\}$

[5] The referee has pointed out that a mean ergodic theorem may be used to prove the following generalization of Theorem B.

*Let $B$ be a reflexive Banach space and let $\{f_n\}$ be a basis in $B$. Suppose $\{g_n\}$ is a sequence in $B$ such that $\|\sum a_n(f_n-g_n)\| < \|\sum a_n f_n\|$ for all $x=\sum a_n f_n$ in $B$. Then the elements $\{g_n\}$ span $B$.*

We have been able to extend his theorem to spaces whose unit sphere has weak sequential compactness.

be a complete orthonormal set. Define $f_n=\psi_{n+1}$ and $g_n=\psi_{n+1}-\psi_n c_{n+1}/c_n$ for $n=1, 2, \cdots$. Thus

$$\|\sum a_n(f_n-g_n)\|^2=\sum\left|\frac{c_{n+1}}{c_n}a_n\right|^2<\sum|a_n|^2,$$

and so relation (4) is satisfied. Obviously the set $\{f_n\}$ is not complete; however $\sum|c_n|^2=\infty$, so the set $\{g_n\}$ is complete.

If the $<$ sign is replaced by the $\leqq$ sign in relation (4), Theorem B is certainly false because some or all of the elements $\{g_n\}$ could be zero. Nevertheless, we are able to prove an analogue of Theorem A.

THEOREM C. *Let $\{g_n\}$ be a complete set and $\{f_n\}$ an orthonormal set such that*

$$\|\sum a_n(f_n-g_n)\|\leqq(\sum|a_n|^2)^{1/2} \tag{5}$$

*for every sequence of constants $\{a_n\}$ such that*

$$0<\sum|a_n|^2<\infty. \tag{6}$$

*Then*

$$0<\|\sum a_n g_n\|\leqq 2(\sum|a_n|^2)^{1/2}. \tag{7}$$

*Conversely, the truth of* (7) *for a complete set $\{g_n\}$ and for every sequence of constants $\{a_n\}$ satisfying* (6) *implies the existence of a (complete) orthonormal set $\{f_n\}$ satisfying* (5).

PROOF. The novelty in the first part of this theorem is the appearance of the strict inequality on the left side of (7). Suppose then, on the contrary, that $\{a_n\}$ is a sequence for which $\sum a_n g_n=0$. Let $\sum a_n f_n=z$. Because $\{g_n\}$ is complete, there exists an element $g_c$ such that $(g_c, z)\neq 0$. Thus, according to (5), if $\lambda$ is a constant:

$$\|\sum a_n(f_n-g_n)+\lambda(f_c-g_c)\|\leqq\|\sum a_n f_n+\lambda f_c\|,$$

$$\|z+\lambda f_c-\lambda g_c\|\leqq\|z+\lambda f_c\|,$$

$$-2\Re\lambda(g_c, z)+|\lambda|^2\{|g_c|^2-2\Re(g_c, f_c)\}\leqq 0.$$

This last inequality is clearly impossible for all values of $\lambda$. The remainder of the proof is omitted as it parallels the proof of Theorem A.

3. **The method of separation of variables.** We prove the following theorem.

THEOREM D. *Let $\{C_{nk}\}$, $n, k=1, 2, \cdots$, be a matrix of constants such that $|C_{nk}|\leqq c_k$. Let $\{T_k\}$ be a sequence of bounded linear trans-*

*formations with corresponding bounds* $\{t_k\}$. *Let* $\{f_n\}$ *be a complete orthonormal set, and define*

$$g_n = f_n + \sum_{k=1}^{\infty} C_{nk} T_k f_n.$$

*Then if* $\sum c_k t_k < 1$, *the set* $\{g_n\}$ *is strongly complete in the sense of Paley and Wiener.*

PROOF. Let $\{a_n\}$ be an arbitrary finite sequence of constants. Then

$$\begin{aligned}
\Big\| \sum a_n (f_n - g_n) \Big\| &= \Big\| \sum_n a_n \sum_k C_{nk} T_k f_n \Big\| \\
&= \Big\| \sum_k T_k \sum_n C_{nk} a_n f_n \Big\| \\
&\leqq \sum_k \Big\| T_k \sum_n C_{nk} a_n f_n \Big\| \\
&\leqq \sum_k t_k \Big\| \sum_n C_{nk} a_n f_n \Big\| \\
&\leqq \sum_k t_k c_k \Big\| \sum_n a_n f_n \Big\|.
\end{aligned}$$

Thus the set $\{g_n\}$ satisfies the Paley-Wiener criterion (2) with $\theta = \sum t_k c_k < 1$.

A simple way to apply Theorem D depends upon the fact that multiplication by a bounded function is a bounded linear transformation of the space $L_2$. In particular we consider the sequence of functions $\{e^{i\lambda_n x}\}$, where $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, is a sequence of complex constants satisfying $|\lambda_n - n| \leqq L$ for some constant $L$. The interval under consideration is $-\pi \leqq x \leqq \pi$. We may write

$$e^{i\lambda_n x} = e^{inx} + \sum_{k=1}^{\infty} \frac{(i\lambda_n - in)^k}{k!} x^k e^{inx}.$$

Comparing with Theorem D, we have:

$$\begin{aligned}
C_{nk} &= (i\lambda_n - in)^k / k!, & c_k &= L^k / k!; \\
T_k &= x^k, & t_k &= \pi^k.
\end{aligned}$$

Clearly, if $L < \log 2/\pi$, the set $\{e^{i\lambda_n x}\}$ is strongly complete in the interval $(-\pi, \pi)$. (Actually the same is true in any interval of length $2\pi$.)

The transformations $\{x^k\}$ never attain their least upper bound. This fact permits Theorem B to be employed in showing that the set $\{e^{i\lambda_n x}\}$ is complete even if $L=\log 2/\pi$. The proof parallels the proof of Theorem D.

The above results on the non-harmonic Fourier series are an extension of previous knowledge in two respects: In the first place, Paley and Wiener were forced to assume that $\{\lambda_n\}$ was a real sequence. Secondly, they[6] obtained the value $1/\pi^2=.10+$ where we have $\log 2/\pi=.22+$. The best value for $L$ is not known; however a theorem of Levinson[7] gives an upper limit of $1/4$.[8]

A second application of Theorem D is to furnish a proof of an analytic function expansion theorem of Boas.[9] In turn, Boas' theorem contains analytic function expansion theorems of Birkhoff, Walsh, Takenaka, G. S. Ketchum and others.

The operator in Theorem D, $\sum C_{nk}T_k$, has been assumed to be a discrete series; however the method of separation of variables is still available if we replace the series by an integral or Stieltjes integral. In particular, Cauchy's integral formula is of the right form.

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PURDUE UNIVERSITY

[6] Paley and Wiener, loc. cit., p. 113. A slightly better value than theirs has been obtained by Malin, thesis, Massachusetts Institute of Technology, 1934.

[7] N. Levinson, Annals of Mathematics, (2), vol. 37 (1936), p. 919; *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, chap. IV.

[8] It is a curious parallelism that $\log 2/\pi$ and $1/4$ are in the same ratio as the limits of Takenaka and Schoenberg in a somewhat similar unsolved problem. For references see R. P. Boas, Jr., Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 139; Transactions of this Society, vol. 48 (1940), p. 485.

[9] R. P. Boas, Jr., Transactions of this Society, vol. 48 (1940), p. 473, Theorem 3.1.

# A GENERALIZATION OF THE POLAR REPRESENTATION OF NONSINGULAR MATRICES

JOHN WILLIAMSON

1. **Introduction.** If $A$ is a square matrix with elements in the complex number field, then

$$A = PU, \tag{1}$$

where $P$ is a positive definite hermitian matrix and $U$ is a unitary matrix.[1] In this polar representation of the matrix $A$, as it is called, the two matrices $P$ and $U$ are unique. Since the matrix $P$ is positive definite and nonsingular, it has the same signature as the identity matrix $E$ while the unitary matrix $U$ is a conjunctive automorph of $E$. From (1) we may deduce a somewhat similar representation of $A$ in terms of a positive definite hermitian matrix and a conjunctive automorph, not of $E$, but of any nonsingular positive definite hermitian matrix.

Let $H$ be a nonsingular hermitian matrix which is positive definite, so that there exists a nonsingular matrix $Q$ satisfying

$$Q^{-1}H(Q^{-1})^* = E,$$

where $Q^*$ is the conjugate transposed of $Q$. If $B = Q^{-1}AQ$ and $B = PU$ is the polar representation of $B$, then $A = QPQ^{-1}QUQ^{-1} = DR$, where $D = QPQ^{-1}$ and $R = QUQ^{-1}$. Since $DH = QPQ^{-1}QQ^* = QPQ^*$, $DH$ is hermitian with the same signature as $H$. Further $RHR^* = QUQ^{-1}QQ^* \cdot (Q^{-1})^*U^*Q^* = QQ^* = H$. Hence we have this result as an analogue of the polar representation (1) of $A$.

RESULT (1). *If $H$ is any nonsingular positive definite hermitian matrix and $A$ is a nonsingular matrix, then*

$$A = DR, \tag{2}$$

*where $DH$ is a positive definite hermitian matrix and $RHR^* = H$.*

If $H = H^{-1}$, $A = DR = DHHR$ and, since $HRH(HR)^* = H^3 = H$, $A = P_1R_1$, where $P_1$ is a positive definite hermitian matrix and $R_1HR_1^* = H$. Therefore we have as a second analogue of (1) the following result.

Received by the editors October 13, 1941.

[1] L. Autonne, *Sur les groupes linéaires, réels et orthogonaux*, Bulletin de la Société Mathématique de France, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

RESULT (2). *If $H$ is a positive definite hermitian matrix, whose square is the identity matrix, and, if $A$ is a nonsingular matrix, then*

$$A = P_1 R_1, \tag{3}$$

*where $P_1$ is a positive definite hermitian matrix and $R_1$ is a conjunctive automorph of $H$. The two matrices $P_1$ and $R_1$ are unique.*

It is our intention here to determine what corresponds to results (1) and (2), if $H$ is a nonsingular hermitian matrix but not necessarily definite.

2. **Generalization in the complex field.** Let $H$ be any nonsingular hermitian matrix and let $A$ be a nonsingular matrix. If

$$A = DR, \tag{4}$$

where

$$DH = HD^* \tag{5}$$

and

$$RHR^* = H, \tag{6}$$

we shall call (4) *a polar representation of $A$ with respect to $H$* or, for brevity an *$H$-representation* of $A$. If (4), (5) and (6) are satisfied, $AHA^*H^{-1} = DRHR^*D^*H^{-1} = DHD^*H^{-1} = D^2$. Further, if $AHA^*H^{-1} = D^2$ and $DH = HD^*$, then (6) is satisfied with $R = D^{-1}A$. For $D^{-1}AH(D^{-1}A)^* = D^{-1}AHA^*(D^{-1})^* = D^{-1}D^2H(D^{-1})^* = H$ by (5). Therefore we have proved this lemma.

LEMMA 1. *A necessary and sufficient condition that a matrix $A$ have an $H$-representation is that there exist a matrix $D$ such that $DH = HD^*$ and $AHA^*H^{-1} = D^2$.*

If $A$ has the $H$-representation (4) and $QAQ^{-1} = A_1$, $QDQ^{-1} = D_1$, $QRQ^{-1} = R_1$ and $QHQ^* = H_1$, then $A_1 = D_1R_1$ where $D_1H_1 = H_1D_1^*$ and $R_1H_1R_1^* = H_1$. Moreover $H_1 = QHQ^*$ and $A_1H_1A_1^* = QAHA^*Q^*$, so that the two pencils of hermitian matrices $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive. Therefore, if $A$ has an $H$-representation and if the two pencils $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive, the matrix $A_1$ has an $H_1$-representation. Accordingly we may suppose that the pencil $AHA^* - xH$ is in a canonical or normal form previously determined.[2] Since $DH$ is hermitian, the elementary di-

[2] G. R. Trott, *On the canonical form of a nonsingular pencil of hermitian matrices*, American Journal of Mathematics, vol. 56 (1934), pp. 359–371. H. W. Turnbull, *Pencils of hermitian forms*, Proceedings of the London Mathematical Society (2), vol. 39 (1935), pp. 232–248.

visors of $D-xE$ are the same as those of the pencil of hermitian matrices $DH-xH$. Complex elementary divisors of such a pencil must therefore occur in conjugate pairs and in particular this is true of the pure imaginary elementary divisors. Hence, if $D-xE$ has the elementary divisor $(x-ib)^r$ repeated $s$ times, where $b$ is real, then $D-xE$ also has the elementary divisor $(x+ib)^r$ repeated $s$ times and $D^2-xE$ has the elementary divisor $(x+b^2)^r$ repeated $2s$ times. Since the elementary divisors of $D^2-xE$ are the same as those of $AHA^*-xH$, the matrix $A$ cannot have an $H$-representation if the pencil $AHA^*-xH$ has an elementary divisor $(x+b^2)^r$, that is, a negative elementary divisor, repeated an odd number of times.

We proceed to determine what further conditions, if any, must be satisfied in order that $A$ may have an $H$-representation. In canonical form the matrices $H$ and $AHA^*$ are similarly partitioned diagonal block matrices of such a nature that it is only necessary to consider the three special cases in which the pencil $AHA^*-xH$ has

(i) only the two elementary divisors $(x-a)^n$, $(x-\bar{a})^n$, $a\neq\bar{a}$,

(ii) the single elementary divisor $(x-b^2)^n$, $b$ real,

(iii) only the two elementary divisors $(x+b^2)^n$, $(x+b^2)^n$, $b$ real.

*Case* (i). In canonical form

$$H=\begin{pmatrix}0 & E\\ E & 0\end{pmatrix} \quad\text{and}\quad AHA^*H^{-1}=\begin{pmatrix}a(E+U) & 0\\ 0 & \bar{a}(E+U)'\end{pmatrix},$$

where $U$ is the auxiliary unit matrix of order $n$. Let

$$G=(E+U)^{1/2}=E+U/2-U^2/8+\cdots+\alpha U^{n-1}, \tag{7}$$

$$2^{2n-4}(n-3)!(n-1)!\alpha=(-1)^n(2n-5)!.$$

Then $G'=(E+U')^{1/2}$. If $a=\rho^2e^{i\theta}$, $a^{1/2}=\delta\rho e^{i\theta/2}$ and $\bar{a}^{1/2}=\delta\rho e^{-i\theta/2}$, where $\delta=\pm1$. Therefore, if $D=[a^{1/2}G,\bar{a}^{1/2}G']$, $DH=HD^*$ and $D^2=AHA^*H^{-1}$. The signature of $DH$ is the same as that of $H$ since the signature of both matrices is zero. Since $\delta$ may have either of the values 1 or $-1$, $D$ is not unique.

*Case* (ii). In canonical form

$$H=\epsilon T \quad\text{and}\quad AHA^*=b^2(E+U)H,$$

where $\epsilon=\pm1$ and $T$ is the counter unit matrix, so that

$$T^2=E \quad\text{and}\quad TU=U'T. \tag{8}$$

If $D=bG$, where $G$ is defined by (7), $D^2=AHA^*H^{-1}$ and $DH=HD^*$ by (8). If $n$ is even the signatures of $DH$ and $H$ are both zero and, if in $DH$, $b$ is replaced by $-b$ the signature of the resulting matrix is

also zero. If $n$ is odd, the signature of $H$ is $\epsilon$ and that of $DH$ is $\pm\epsilon$ according as $b$ is positive or negative. Therefore, if $DH$ has the same signature as $H$ and $n$ is even, $D$ is not unique but, if $n$ is odd, $D$ is unique.

*Case* (iii). If the matrix $D$ exists the pencil $DH-xH$ has only the two elementary divisors $(x+ib)^n$, $(x-ib)^n$ and we may take $DH-xH$ in the canonical form

$$H=\begin{pmatrix}0 & E\\ E & 0\end{pmatrix},\qquad D=\begin{pmatrix}ibG & 0\\ 0 & -ibG'\end{pmatrix}, \tag{9}$$

where $G$ is defined by (7). Hence

$$AHA^*H^{-1}=D^2=\begin{pmatrix}-b^2(E+U) & 0\\ 0 & -b^2(E+U')\end{pmatrix}$$

and

$$H=\begin{pmatrix}0 & E\\ E & 0\end{pmatrix},$$

while the matrix $H$ may be transformed without disturbing $D^2$ into

$$\begin{pmatrix}T & 0\\ 0 & -T\end{pmatrix}.$$

For a matrix pencil $AHA^*-xH$ with elementary divisors $(x+b^2)^n$, $(x+b^2)^n$ there are three distinct possible canonical forms in which $H$ is one of the matrices

$$\begin{pmatrix}\epsilon_1 T & 0\\ 0 & \epsilon_2 T\end{pmatrix},$$

$\epsilon_1=1$, $\epsilon_2=-1$; $\epsilon_1=\epsilon_2=1$ or $\epsilon_1=\epsilon_2=-1$. If we call $\epsilon_1$ and $\epsilon_2$ the *indices* associated with the elementary divisor $(x+b^2)^n$, we see that $A$ has an $H$-representation, if and only if one index is positive and the other is negative. Further, even when $A$ does have an $H$-representation, $D$ is not unique. Since the canonical form for the pencil $AHA^*-xH$ is diagonal block, the following theorem follows immediately from Lemma 1.

THEOREM 1. *Let $H$ be any nonsingular hermitian matrix and $A$ be a nonsingular matrix. Then $A$ has an $H$-representation $A=DR$, if and only if the negative elementary divisors of the pencil $AHA^*-xH$ occur in pairs and exactly half of the indices associated with each negative elementary divisor are positive. The matrix, $D$, and therefore the matrix*

*R, is unique, if and only if all elementary divisors of the pencil are positive and of odd order.*

If $H^2=E$, $A=DHHR=SV$, where $S=DH$ and $V=HR$. Moreover $S=S^*$ and $VHV^*=H^2=H$. Hence we have as a corollary the following.

COROLLARY 1. *If $H^2=E$, the matrix $A$ can be written in the form $A=SV$, where $S$ is hermitian with the same signature as $H$ and, $VHV^*=H$, if and only if the conditions of Theorem 1 are satisfied.*

The known theorem on the uniqueness of the polar components of the matrix $A$ in (1) is a particular case of Theorem 1. For, if $H=E$, $AHA^*=AA^*$ and the elementary divisors of the pencil $AA^*-xE$ are all positive and linear.

3. **Modified representation of any nonsingular matrix.** Even if the matrix $A$ does not satisfy the conditions of Theorem 1, it is possible to find a somewhat different representation of $A$. We first reduce the pencil $AHA^*-xH$ by conjunctive transformation to

$$\begin{pmatrix} A_1H_1A_1^* & 0 \\ 0 & A_2H_2A_2^* \end{pmatrix} - x\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where no elementary divisor of $A_1H_1A_1-xH_1$ is negative and all elementary divisors of $A_2H_2A_2^*-xH_2$ are negative. Then $A_1=D_1R_1$ and, since all elementary divisors of $-A_2H_2A_2^*-xH_2$ are positive, there exists a matrix $D_2$, such that $D_2^2=-A_2H_2A_2^*H_2^{-1}$ and $D_2H_2=H_2D_2^*$. If $R_2=D_2^{-1}A_2$, then $R_2H_2R_2^*=D_2^{-1}A_2H_2A_2^*(D_2^*)^{-1}=D_2\cdot(-D_2^2H_2)(D_2^*)^{-1}=-H_2$. Therefore $A_2=D_2R_2$, where $D_2H_2$ is hermitian with the same signature as $H_2$ and $R_2H_2R_2^*=-H_2$. Let

$$V=\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix}=[E_1, -E_2],$$

where $E_i$ is the unit matrix of the same order as $H_i$. Then $V^2=E$. If $A=[A_1, A_2]$, $H=[H_1, H_2]$, $R=[R_1, R_2]$ and $D=[D_1, D_2]$, then $A=DR$ where $DH$ is hermitian with the same signature as $H$ and $RHR^*=HV$, so that the signature of $HV$ is the same as that of $H$. Further from its form it is apparent that $V$ is a polynomial in $AHA^*H^{-1}$ and that $VH=HV=HV^*$, so that $VHV^*=H$. While we have used a special form of the pencil $AHA^*-xH$ it follows that similar results are true when the pencil is not in this form. Accordingly we have proved this theorem.

THEOREM 2. *Let $H$ be a nonsingular hermitian matrix and let $A$ be*

*a nonsingular matrix of the same order as $H$. Then there exists at least one matrix $V$, a polynomial in $AHA^*H^{-1}$, with the properties that $V^2=E$, $VH$ is hermitian with the same signature as $H$ and that no characteristic number of $AHA^*(VH)^{-1}$ is negative. The matrix $A=DR$ where $DH$ is hermitian with the same signature as $H$ and $RHR^*=VH$. If all the characteristic numbers of $AHA^*H^{-1}$ are real, the matrix $V$ is uniquely determined.*

4. **Representation in the real field.** If $H$ is a real symmetric matrix and $A$ a real nonsingular matrix our argument may be carried through in the field of real numbers with only a few alterations. In (i), if the pair of conjugate elementary divisors is $(x-c\pm id)^n$, we take $H=qT$, $AHA^*=p(E+U)qT$, where

$$p=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad \text{and} \quad q=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, if

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}^2 = p,$$

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix} GH$$

is symmetric and

$$\left\{\begin{pmatrix} r & s \\ -s & r \end{pmatrix} G\right\}^2 = AHA^*H^{-1}.$$

Case (ii) is unaltered. In (iii) we replace the matrices $D$ and $H$ of (9) by

$$\begin{pmatrix} 0 & bG \\ -bG & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix},$$

respectively. It therefore follows that Theorems 1 and 2 are true in the field of real numbers, if hermitian is replaced by symmetric and conjugate transposed by transposed.

5. **Conditions for the commutativity of the polar components.** Let the matrix $A$ have an $H$-representation $A=DR$. Then $A=RD_1$ where $D_1=R^{-1}DR$. The matrix $D_1H$ is hermitian. Further $HA^*H^{-1}A=HD_1^*R^*H^{-1}RD_1=D_1HH^{-1}D_1=D_1^2$. Since $D^2=AHA^*H^{-1}$, it follows that, if $A^*$ is commutative with $H^{-1}AH$, $D_1^2=D^2$ and that for a proper choice $D_1=D$. Conversely, if $D_1=D$, $H^{-1}AH$ is commutative with $A^*$. We have therefore proved this theorem.

THEOREM 3. *If $A$ has an $H$-representation $A=DR$, $A$ also has an $H$-representation $A=RD_1$. A necessary and sufficient condition that $D=D_1$ or that $D$ be commutative with $R$ is that $A^*$ be commutative with $H^{-1}AH$.*

Therefore, if $A=DR$ and $A$ is normal[3] with respect to $H$, $A=RD$ so that $R$ and $D$ are commutative. That the converse is not true may be shown as follows. Let

$$H=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \quad \text{and} \quad A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

Then

$$AHA^*H^{-1}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\1&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&0\\0&1&0&0\end{pmatrix}=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

The characteristic numbers of $AHA^*H^{-1}$ are all plus one and therefore $A$ has an $H$-representation. In fact

$$D=\begin{pmatrix}1&1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix} \quad \text{and} \quad R=\begin{pmatrix}1&-1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix}.$$

Hence

$$H^{-1}AH=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and is commutative with $A^*$. However, $H^{-1}AH$ is not a polynomial in $A^*$ and is therefore not normal with respect to $H$.

If the elementary divisors of $AHA^*-xH$ are all positive and linear and if $A=DR=RD$, it is comparatively easy to show that $H^{-1}AH$ is a polynomial in $A^*$ and therefore that $A$ is normal with respect to $H$.

[3] John Williamson, *Matrices normal with respect to an hermitian matrix*, American Journal of Mathematics, vol. 60 (1938), pp. 355–373.

This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix $A$ to be commutative is that $A$ be a normal matrix.

QUEENS COLLEGE

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## REMARKS ON REGULARITY OF METHODS OF SUMMATION

G. E. FORSYTHE AND A. C. SCHAEFFER

A doubly infinite matrix[1] $(a_{mn})$ $(m, n = 1, 2, \cdots)$ is said to be *regular*, if for every sequence $x = \{x_n\}$ with limit $x'$ the corresponding sums $y_m = \sum_{n=1}^{\infty} a_{mn}x_n$ exist for $m = 1, 2, \cdots$, and if $\lim_{m\to\infty} y_m = x'$. An apparently more inclusive definition of regularity is that for each sequence $x$ with limit $x'$ the sums defining $y_m$ shall exist for all $m \geqq m_0(x)$ and $\lim_{m\to\infty} y_m = x'$. Tamarkin[2] has shown that $(a_{mn})$ is regular in the latter sense if and only if there exists an $m_1$ *independent of* $x$ such that the matrix $(a_{mn})$ $(m \geqq m_1,\ n \geqq 1)$ is regular in the former sense. Using point set theory in the Banach space $(c)$, he proves a theorem[3] from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

THEOREM 1. *Suppose the doubly infinite matrix* $(a_{mn})$ *has the property that for each sequence* $x = \{x_n\}$ *with limit* 0 *there exists an* $m_0 = m_0(x)$ *such that for all* $m \geqq m_0(x)$, $u_m = \limsup_{k\to\infty} \left|\sum_{n=1}^{k} a_{mn}x_n\right| < \infty$. *Then there exists an* $m_1$ *such that* $\sum_{n=1}^{\infty} |a_{mn}| < \infty$ *for all* $m \geqq m_1$.

If in addition $\lim_{m\to\infty} u_m = 0$ for each sequence $x$ with limit 0, it will follow[4] that there exists an $N$ such that $\sum_{n=1}^{\infty} |a_{mn}| \leqq N < \infty$, for all $m \geqq m_1$.

To prove Theorem 1, suppose there were an infinite sequence $m_1 < m_2 < \cdots$ such that $\sum_{n=1}^{\infty} |a_{mn}| = \infty$ for $m \in \{m_r\}$. Let $x_1, \cdots, x_{k_1}$ be chosen with unit moduli and with amplitudes such that

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Presented to the Society, April 11, 1942 under the title *A remark on Toeplitz matrices*; received by the editors January 22, 1942.

[1] In this note $a_{mn}$, $x_n$ and $x'$ denote finite complex numbers.

[2] J. D. Tamarkin, *On the notion of regularity of methods of summation of infinite series*, this Bulletin, vol. 41 (1935), pp. 241–243.

[3] J. D. Tamarkin, loc. cit., p. 242, lines 1–6.

[4] See, for example, I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, vol. 151 (1921), pp. 79–111; p. 85, Theorem 4.

$$\sum_{n=1}^{k_1} a_{m_1 n} x_n = \sum_{n=1}^{k_1} |\, a_{m_1 n} x_n \,| > 1.$$

Let $x_{k_1+1}, \cdots, x_{k_2}$ be chosen with moduli $1/2$ and with amplitudes such that

$$\left| \sum_{n=k_1+1}^{k_2} a_{m_2 n} x_n \right| > 2 + \sum_{n=1}^{k_1} |\, a_{m_2 n} x_n \,|.$$

Let $x_{k_2+1}, \cdots, x_{k_3}$ be chosen with moduli $1/3$ and with amplitudes such that

$$\left| \sum_{n=k_2+1}^{k_3} a_{m_3 n} x_n \right| > 3 + \sum_{n=1}^{k_2} |\, a_{m_3 n} x_n \,|.$$

Writing $y_m(k) = \sum_{n=1}^{k} a_{mn} x_n$, the sequence $\{x_n\}$ and integers $k_1 < k_2 < \cdots$ are thus chosen successively so that $|y_{m_1}(k_1)| > 1$, $|y_{m_2}(k_2)| > 2$; $|y_{m_1}(k_3)| > 3$, $|y_{m_2}(k_4)| > 4$, $|y_{m_3}(k_5)| > 5$; $|y_{m_1}(k_6)| > 6$, $\cdots$; while $|x_n| = 1/r$, for $k_{r-1} < n \leqq k_r$. This is a sort of alternating or "sweeping-out" process. So defined, $\{x_n\}$ is a sequence with limit 0, but $\limsup_{k\to\infty} |\sum_{n=1}^{k} a_{mn} x_n| = \infty$, for $m \in \{m_r\}$. This contradiction completes the proof of Theorem 1.

The matrix $(a_{mn})$ is said to be *null-preserving*, if for every sequence $x = \{x_n\}$ with limit 0 the corresponding sums defining $y_m$ exist for $m = 1, 2, \cdots$ and if $\lim_{m\to\infty} y_m = 0$. An apparently more inclusive definition of null-preserving is that for each sequence $x$ with limit 0 we have $u_m = \limsup_{k\to\infty} |\sum_{n=1}^{k} a_{mn} x_n| < \infty$ for all $m \geqq m_0(x)$ and $\lim_{m\to\infty} u_m = 0$. We remark that it is a consequence of Theorem 1 that $(a_{mn})$ is null-preserving in the latter sense if and only if there exists an $m_1$ such that the matrix $(a_{mn})$ $(m \geqq m_1, n \geqq 1)$ is null-preserving in the former sense.[5]

To consider a problem which is related to the above in the method of proof, let each element of a matrix $(a_{mn})$ be either $+1$ or $-1$. For $0 \leqq t \leqq 1$ and $n = 1, 2, \cdots$ let $\{\phi_n(t)\}$ be the Rademacher orthogonal functions,[6] and let $y_{mk}(t) = \sum_{n=1}^{k} a_{mn} \phi_n(t)$. Then it is well known[7] that for almost all $t$, for all $m = 1, 2, \cdots$ and for all $\epsilon > 0$, $\lim_{k\to\infty} k^{-1/2-\epsilon} y_{mk}(t) = 0$. It is clear that for a particular fixed $m$ there is a $t$ such that $\lim_{k\to\infty} k^{-1} y_{mk}(t) = 1$. The problem is to show that there is a $t$ such that

[5] For conditions that $(a_{mn})$ be null-preserving, see T. Kojima, *On generalized Toeplitz's theorems on limit and their applications*, Tôhoku Mathematical Journal, vol. 12 (1917), pp. 291–326; p. 300.

[6] A. Zygmund, *Trigonometrical Series*, Warsaw, 1935, p. 5.

[7] For references to this and more precise results, see A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, Berlin, 1933, pp. 60–61.

simultaneously for all $m=1, 2, \cdots$, $\limsup_{k\to\infty} k^{-1}y_{mk}(t)=1$. That there exists such a $t$ can be shown by using the alternating process of Theorem 1.

Theorem 2 follows immediately from a theorem of Banach.[8]

THEOREM 2. *If $E_m$ is a linear manifold satisfying Baire's condition*[9] *in a Banach space $E$ $(m=1, 2, \cdots)$ and if $\lim_{m\to\infty} E_m=E$, then there exists an $m_1$ such that $E_{m_1}=E$.*

Theorem 2 furnishes a Banach space analogue and a proof of Theorem 1 which is related to Tamarkin's proof. To see this, let $E$ be the Banach space $(c_0)$ of sequences $x=\{x_n\}$ convergent to 0, with $\|x\|=\max_n |x_n|$, and with addition and multiplication by a (complex) scalar defined as usual. Let $(a_{mn})$ be as in Theorem 1. Let $E_m$ be the subset of $E$ for which $\limsup_{k\to\infty}|\sum_{n=1}^{k} a_{rn}x_n| < \infty$ for all $r\geqq m$. The hypotheses of Theorem 2 are satisfied, and from its conclusion it may be proved directly for an arbitrary $m\geqq m_1$ that $\sum_{n=1}^{\infty}|a_{mn}| < \infty$.

STANFORD UNIVERSITY

[8] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 22, Theorem 2.

[9] See S. Banach, loc. cit., p. 17. By considering a Hamel base for $E$, G. W. Mackey has remarked to the authors that Theorem 2 is false if the words "satisfying Baire's condition" are omitted.

# PROJECTIONS OF THE PRIME-POWER ABELIAN GROUP OF ORDER $p^m$ AND TYPE $(m-1, 1)$

ROSS A. BEAUMONT

1. **Introduction.** A function $f$ of the subgroups of the group $G$ upon the subgroups of the group $H$ is called a projectivity of $G$ upon $H$ ($f(G)=H$) if the following hold.

(1) For every subgroup $S$ of $G$, $f(S)$ is a subgroup of $H$.

(2) If $S'$ is a subgroup of $H$, then there exists a subgroup $S$ of $G$ such that $f(S)=S'$.

(3) If $S$ and $T$ are subgroups of $G$, $S \leqq T$ is a necessary and sufficient condition that $f(S) \leqq f(T)$.

The correspondence $f$ is a (1-1) correspondence which preserves the partial ordering of the set of subgroups of the group $G$.

Further, a projectivity $f$ is called index-preserving if $[T:S] = [f(T):f(S)]$ for subgroups $S$ of cyclic subgroups $T$ of $G$; and $f$ is called strictly index-preserving if $[T:S]=[f(T):f(S)]$ for subgroups $S$ of subgroups $T$ of $G$.

If $G$ is the direct product of cyclic groups of order $p$, $p$ a prime number, R. Baer[1] has given necessary and sufficient conditions that a group $H$ be a projection of $G$. In particular he has shown that if the projectivity of $G$ upon $H$ is index-preserving, then $G$ and $H$ are isomorphic. Thus in a study of the projections of the prime-power abelian group of order $p^m$ and type $(m-1, 1)$, we need consider only the case $m>2$.

Rottlaender[2] investigated the case $m=3$ and found necessary and sufficient conditions for the existence of a strictly index-preserving projectivity of the prime-power abelian group $G$ of order $p^3$ and type $(2, 1)$ upon a group $H$.

In this note, Baer's general results are used to find the necessary and sufficient conditions for the existence of a projectivity of the prime-power abelian group $G$ of order $p^m$ and type $(m-1, 1)$ upon a group $H$.

2. **The necessary conditions.** If $G$ is an abelian group of the type under consideration, $G=\{u_1\} \times \{u_2\}$ where $u_1$ is of order $p^{m-1}$, $m>2$,

Presented to the Society, April 11, 1942; received by the editors January 19, 1942.

[1] R. Baer, *The significance of the system of subgroups for the structure of the group*, American Journal of Mathematics, vol. 61 (1939), pp. 1–44. Hereafter this paper will be referred to as B.

[2] Ada Rottlaender, *Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 641–653.

and $u_2$ is of order $p$. If $f(G)=H$, it follows from known results[3] that $f$ is index-preserving, and hence $H=\{f(\{u_1\}), f(\{u_2\})\}$ where $f(\{u_1\})=\{u_1'\}$ is a cyclic group of order $p^{m-1}$, $f(\{u_2\})=\{u_2'\}$ is a cyclic group of order $p$. Thus $H=\{u_1', u_2'\}$ where $u_1'$ and $u_2'$ are independent generators of order $p^{m-1}$ and $p$, respectively.

If $K$ is any group, then $K^p$ is the set of all $p$th powers of the elements of $K$, and $K^p$ is a characteristic subset of $K$.

LEMMA 1. *$f(G^p)=H^p$ is a characteristic subgroup of $H$.*

PROOF. $G^p=\{u_1^p\}$ and $f(G^p)=f(\{u_1^p\})=f(\{u_1\})^p=\{u_1'^p\}$ since $f$ is index-preserving. $\{u_1'^p\}\leqq H^p$. If $x$ is in $H^p$, $x=y^p$ where $y$ is in $H$. $f^{-1}(\{x\})=f^{-1}(\{y^p\})=f^{-1}(\{y\})^p=\{u_1^k u_2^n\}^p=\{u_1^{kp}\}$. $f(\{u_1^{kp}\})=f(\{u_1\})^{kp}=\{u_1'^{kp}\}=\{x\}$. Hence $x=u_1'^{ikp}=(u_1'^p)^{ik}$ and $x$ is in $\{u_1'^p\}$. Thus $H^p\leqq\{u_1'^p\}$ so that we have shown that $f(G^p)=\{u_1'^p\}=H^p$. Since $H^p$ is a subgroup of $H$, it is a characteristic subgroup, and in particular normal.

Since $f(G)=H$ and $f(G^p)=H^p$, the index-preserving projectivity $f$ of $G$ upon $H$ induces an index-preserving projectivity of $G/G^p$ upon $H/H^p$. $G/G^p$ is the direct product of two cyclic groups of order $p$ and hence by the result[4] mentioned above, $G/G^p$ and $H/H^p$ are isomorphic. Thus $H/H^p$ is abelian and $H^p$ contains the commutator subgroup of $H$. This implies

$$u_2'^{-1}u_1'u_2' = u_1'^{1+jp}. \tag{1}$$

The following multiplication rules for elements in $H$ follow from (1)

$$u_2'^{-1}u_1'^{h}u_2' = u_1'^{h(1+jp)}, \tag{2}$$

$$u_1'^{h}u_2'^{i} = u_2'^{i}u_1'^{h(1+jp)^i}, \tag{3}$$

$$\begin{aligned}(u_2'^{i}u_1'^{h})^k &= u_2'^{ki}u_1'^{h[1+(1+jp)^i+\cdots+(1+jp)^{(k-1)i}]} \\ &= u_2'^{ki}u_1'^{h[(1+jp)^{ki}-1]/[(1+jp)^i-1]}\end{aligned} \tag{4}$$

It follows from (3) that the order of $H$ is $p^m$ so that we have proved the following theorem.

THEOREM 1. *If $G$ is the prime-power abelian group of order $p^m$, $m>2$, and type $(m-1, 1)$ and if $f$ is a projectivity of $G$ upon a group $H$, then $H$ is a prime-power group of order $p^m$ generated by independent generators $u_1'$ of order $p^{m-1}$ and $u_2'$ of order $p$ such that $u_2'^{-1}u_1'u_2'=u_1'^{1+jp}$.*

If $p=2$, we derive the additional necessary condition.

[3] [B, Corollary 11.3].

[4] [B, Corollary 8.2].

LEMMA 2. $j \equiv 0 \bmod 2$ *so that* $u_2'^{-1}u_1'u_2' = u_1'^{1+4w}$.

PROOF. Since $f(\{u_1\}) = \{u_1'\}$, it follows from Baer's results[5] that there exists one and only one element $u'$ in $H$ such that $f(\{u_2\}) = \{u'\}$ and $f(\{u_2u_1\}) = \{u'u_1'\}$. Since $\{u'\} = \{u_2'\}$, $u' = u_2'^{\rho}$ where $\rho$ is odd. $f(\{u_2u_1\}^2) = f(\{u_2u_1\})^2$ since $f$ is index-preserving and we have

$$f(\{u_2u_1\}^2) = f(\{u_1^2\}) = f(\{u_1\})^2 = \{u_1'^2\},$$
$$f(\{u_2u_1\})^2 = \{(u'u_1')^2\} = \{(u_2'^{\rho}u_1')^2\}$$
$$= \{u_2'^{2\rho}u_1'^{[(1+2j)^{2\rho}-1]/[(1+2j)^{\rho}-1]}\} = \{u_1'^{(1+2)^{\rho}+1}\}.$$

Thus $u_1'^{(1+2j)^{\rho}+1} = u_1'^{2\gamma}$ where $\gamma$ is odd and we have

$$(1+2j)^{\rho} + 1 \equiv 2\gamma \bmod 2^{m-1},$$
$$2 + 2\rho j + [\rho(\rho-1)/2](2j)^2 + \cdots \equiv 2\gamma \bmod 2^{m-1},$$
$$1 + \rho j + [\rho(\rho-1)/2]2j^2 + \cdots \equiv \gamma \bmod 2^{m-2}$$

and recall that $m > 2$. Since $\gamma$ is odd, the left member of the congruence is odd which implies $\rho j$ is even. Since $\rho$ is odd, $j$ is even, which completes the proof of the lemma.

3. **Construction of a projectivity for groups satisfying the necessary conditions.** If $p$ is an odd prime and $H$ is a group satisfying the necessary conditions, then $H$ is either abelian or $H$ is the unique non-abelian group $\{U_1, U_2\}$ where $U_1$ and $U_2$ are subject to the sole defining relations[6]

$$U_1^{p^{m-1}} = U_2^p = 1, \qquad U_2^{-1}U_1U_2 = U_1^{1+p^{m-2}}. \tag{5}$$

If $H$ is abelian, then $G$ and $H$ are isomorphic and this isomorphism induces a projectivity of $G$ upon $H$.

If $H$ is non-abelian we will construct a projectivity of $G$ upon $H$ by establishing a correspondence between the cyclic subgroups and then extending this correspondence to a projectivity.

From (5) we find the following multiplication rule for elements of $H$

$$(U_2^iU_1^k)^h = U_2^{ih}U_1^{k[h+ik(k-1)p^{m-2}/2]}. \tag{6}$$

Every element of $G$ has the unique form $u_2^su_1^r$ where $0 \leqq s < p$, $0 \leqq r < p^{m-1}$. If $s > 0$, there exists an integer $x$, uniquely determined

[5] [B, (9.2), (d)].

[6] Carmichael, *Introduction to the Theory of Groups of Finite Order*, p. 132.

mod $p^{m-1}$, such that $sx \equiv 1 \bmod p^{m-1}$, $(x, p) = 1$. Then $u_1^r = (u_1^r)^{sx} = u_1^{rx} = u_1^{ts}$ where $rx = t$. $u_2^s u_1^r = u_2^s u_1^{ts}$ and $\{u_2^s u_1^r\} = \{(u_2 u_1^t)^s\} = \{u_2 u_1^t\}$ where $t$ is uniquely determined mod $p^{m-1}$. We define the following correspondence of the cyclic subgroups of $G$ upon those of $H$

$$f(\{u_2 u_1^t\}) = \{U_2 U_1^t\}, \tag{7}$$

$$f(\{u_1^r\}) = \{U_1^r\}. \tag{8}$$

The correspondence $f$ is a (1-1) correspondence of the set of cyclic subgroups of $G$ upon a subset of the set of cyclic subgroups of $H$. To show that $f$ is a (1-1) correspondence on the whole set of cyclic subgroups of $H$, it is only necessary to show that every cyclic subgroup of $H$ has the form $\{U_2 U_1^t\}$ or $\{U_1^r\}$ where $t$ or $r$ is uniquely determined mod $p^{m-1}$, respectively.

Every element of $H$ has the unique form $U_2^s U_1^r$ where $0 \leqq s < p$, $0 \leqq r < p^{m-1}$. If $s > 0$, there exists an integer $x$, uniquely determined mod $p^{m-1}$, such that $sx \equiv 1 \bmod p^{m-1}$, $(x, p) = 1$. Then

$$\begin{aligned} \{U_2^s U_1^r\} &= \{(U_2^s U_1^r)^x\} \\ &= \{U_2^{sx} U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\} \\ &= \{U_2 U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\} \end{aligned}$$

where $r[x+(sx(x-1)/2)p^{m-2}]$ is uniquely determined mod $p^{m-1}$.

The correspondence $f$ preserves the indices of the cyclic subgroups since

$$f(\{u_2 u_1^t\}^p) = f(\{u_1^{tp}\}) = \{U_1^{tp}\}$$

by (8),

$$f(\{u_2 u_1^t\})^p = \{U_2 U_1^t\}^p = \{U_1^{t[p+(p(p-1)/2)p^{m-2}]}\} = \{U_1^{tp}\},$$

by (7) and (6).

Since the only non-cyclic subgroups of $H$ are those of the form $\{U_1^{p^\lambda}, U_2\}$, $0 \leqq \lambda < m-1$, by extending $f$ so that

$$f(\{u_1^{p^\lambda}, u_2\}) = \{U_1^{p^\lambda}, U_2\} \tag{9}$$

$f$ becomes an index-preserving projectivity of $G$ upon $H$.

Combining the above results with Theorem 1 we have this theorem.

THEOREM 2. *If $G$ is the prime-power abelian group of odd order $p^m$, $m > 2$, and type $(m-1, 1)$ then there exists a projectivity $f$ of $G$ upon a group $H$ if, and only if, either $H$ is isomorphic to $G$ or $H$ is the non-*

*abelian group* $\{U_1, U_2\}$ *where* $U_1$ *and* $U_2$ *are subject to the sole defining relations* (5).

If $p=2$ it follows from Lemma 2 that if $m=3$, then $H$ is abelian and hence $G$ and $H$ are isomorphic groups. If $m>3$, it follows from Lemma 2 and from known results[7] that $H$ is either abelian or the non-abelian group $\{U_1, U_2\}$ where $U_1$ and $U_2$ are subject to the sole defining relations

$$U_1^{2^{m-1}} = U_2^2 = 1, \qquad U_2 U_1 U_2 = U_1^{1+2^{m-2}} . \tag{10}$$

Since Baer has shown[8] that there exists an index-preserving projectivity of $G$ upon this non-abelian group we have the following theorem.

THEOREM 3. *If* $G$ *is the prime-power abelian group of order* $2^m$, $m>2$, *and type* $(m-1, 1)$ *then*

(a) *if* $m=3$, *there exists a projectivity* $f$ *of* $G$ *upon a group* $H$ *if and only if* $G$ *and* $H$ *are isomorphic groups*;

(b) *if* $m>3$, *there exists a projectivity* $f$ *of* $G$ *upon a group* $H$ *if, and only if, either* $G$ *and* $H$ *are isomorphic groups or* $H$ *is the non-abelian group* $\{U_1, U_2\}$ *where* $U_1$ *and* $U_2$ *are subject to the sole defining relations* (10).

UNIVERSITY OF WASHINGTON

[7] Carmichael, loc. cit., p. 133.

[8] [B, p. 11].

# SYMMETRIC DIFFERENTIAL EXPRESSIONS

HENRY J. RIBLET

**1. Introduction.** In a recent article,[1] it is shown that if

$$(1) \qquad f(x_1, \cdots, x_n) = \sum c_{\nu_{01}, \cdots, \nu_{kn}} x_1^{\nu_{10}} \cdots (x_1^{(1)})^{\nu_{11}} \cdots (x_n^{(k)})^{\nu_{kn}},$$

where $x_i^{(k)}$ represents the $k$th derivative of $x_i$ and $f(x_1, \cdots, x_n)$ is unchanged by all permutations of the variables, then

$$(2) \qquad f(x_1, \cdots, x_n) = \frac{1}{D^p} \sum C_{\lambda_{01}, \cdots, \lambda_{kn}} a_1^{\lambda_{10}} \cdots (a_1^{(1)})^{\lambda_{11}} \cdots (a_n^{(k)})^{\lambda_{kn}}$$

where $D$ is the discriminant of $x_1, \cdots, x_n$, and $a_1, \cdots, a_n$ are the elementary symmetric functions or E.S.F.'s. In applications to problems involving the differential equations satisfied by algebraic functions or the algebraic properties of the solutions of algebraic differential equations, it is desirable to have some method for passing from (1) to (2). The proof of the basic theorem, although constructive, gives a method that is prohibitively laborious and will often introduce unnecessary powers of $D$. It is the object of this paper to present an exhaustion procedure which greatly simplifies the work and obviates the latter danger.

**2. Preliminaries.** It is no restriction to limit ourselves to symmetric differential functions generated by a single term of (1). Term $A$ of such a function will be said to be of higher *order* than term $B$ if the first exponent $\nu_{ij}$ in $A$ which differs from the corresponding exponent $\nu'_{ij}$ in $B$ is the larger.

THEOREM 1. *The exponents of any power product of the derivatives of the E.S.F.'s are uniquely determined by the highest order term; and the exponents $\nu_{ij}$ of the highest order term satisfy the inequalities, where $j=1, \cdots, n-1$,*

$$(3) \qquad \nu_{0j} - \nu_{0j+1} \geqq \sum_{i=1}^{k} \nu_{ij+1}.$$

PROOF. It is clear that the highest order term in $a_i^{(j)}$ is $x_1 \cdots x_{i-1} x_i^{(j)}$ and that the highest order term in

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Presented to the Society, October 28, 1939 under the title *Symmetric differential expressions and applications*; received by the editors January 31, 1942.

[1] H. J. Riblet, *Algebraic differential fields*, American Journal of Mathematics, vol. 63 (1941), p. 341.

$$a_1^{\lambda_{01}} \cdots a_n^{\lambda_{0n}} (a_1^{(1)})^{\lambda_{11}} \cdots (a_n^{(k)})^{\lambda_{kn}}$$

is

$$x_1^{\nu_{01}} \cdots x_n^{\nu_{0n}} (x_1^{(1)})^{\nu_{11}} \cdots (x_n^{(k)})^{\nu_{kn}}$$

where we have $\nu_{ij}=\lambda_{ij}$ for $i$ greater than zero and otherwise

$$\nu_{0l} = \lambda_{0l} + \sum_{i,j>l} \lambda_{ij}.$$

Now

$$\begin{aligned} \nu_{0l} - \nu_{0l+1} &= \lambda_{0l} + \sum_{i,j>l} \lambda_{ij} - \left(\lambda_{0l+1} + \sum_{i,j>l+1} \lambda_{ij}\right) \\ &= \lambda_{0l} + \sum_{i=1}^{k} \lambda_{il+1} = \lambda_{0l} + \sum_{i=1}^{k} \nu_{il+1} \end{aligned} \tag{4}$$

and since $\lambda_{0l} \geqq 0$ condition (3) is certainly satisfied. Moreover, if it is satisfied, the equations $\nu_{ij}=\lambda_{ij}$, for $i$ greater than one and (4) otherwise determine all the $\lambda$'s uniquely.

3. **The procedure.** We now consider the possibility of finding expression (2) by subtracting from $f(x_1, \cdots, x_n)$ a suitable multiple of that power product of the derivatives of the E.S.F.'s which its highest order term defines, by Theorem 1; and then proceeding in a similar manner with the resulting function.

THEOREM 2. *A necessary and sufficient condition that* (1) *be expressible as a rational integral function of the E.S.F.'s and their derivatives is that the highest order term of each function obtained by successively eliminating highest order terms satisfy condition* (3).

PROOF. From Theorem 1 it follows that condition (3) is certainly necessary. Sufficiency also follows from this theorem if we may be assured that there are only a finite number of terms lower than the original highest order term. This follows from the fact that in each term which occurs $\sum i\nu_{ij}=\sum i\lambda_{ij}=M$, a fixed quantity.

If the conditions of Theorem 2 are not satisfied, we will encounter a symmetric differential function $f_2$ whose highest order term does not satisfy condition (3). The highest order term of $D$ is $x_1^{2n-2}x_2^{2n-4} \cdots x_n^2$. It is clear that multiplication of $f_2$ by a sufficiently high power $p_2$ of $D$ will give a function whose highest order term satisfies condition (3). Thus we can continue as before hoping that the process will terminate.

THEOREM 3. *The exhaustion procedure terminates and when it does it gives the minimum power of D for which an expression of the form of* (2) *exists.*

PROOF. Let $p$ be the smallest power of $D$ for which an expression of the form (2) exists, and suppose that in the exhaustion process we have reached a point so that

$$(5) \qquad f(x_1, \cdots, x_n) = f_1 + \frac{f_2}{D^{p_2}} + \frac{f_3}{D^{p_3}} + \cdots + \frac{f_r + f_r'}{D^{p_r}},$$

where $f_i$ denotes a rational integral function of the E.S.F.'s and their derivatives, $p_r \geqq p$, and $b_r$, the highest order term of $f_r'$, does not satisfy condition (3). If we replace $f(x_1, \cdots, x_n)$ by its equivalent as given by equation (2) and multiply both sides of the result by $D^{p_r}$, we obtain an expression for $f_r'$ as a rational, integral function of the E.S.F.'s and their derivatives. This however contradicts Theorem 1 unless $b_r = 0$ and this implies that $f_r' \equiv 0$ which in turn proves the first assertion. Now if $p_r > p$, we can immediately prove that $D$ divides $f_r$ considered as a function of the E.S.F.'s and their derivatives. This completes the proof of the theorem.

As a corollary of Theorem 2 and as an example of the fact that the power of $D$ occurring in the denominator of (2) may be less, for an extensive class of functions, than might be expected from the original construction, we prove this theorem:

THEOREM 4. *An integral rational symmetric differential expression which is linear in its differentiated terms may be expressed as an integral rational function of the E.S.F.'s and their first derivatives.*

PROOF. It is clear that the theorem is proved, if we can show its truth for the symmetric function generated by the term

$$\sum_{i=1}^{n} x_i^{a_i} x_j^{(1)}.$$

Now the highest order term for any such differential expression satisfies condition (3), for, if not, $a_{j-1} \geqq a_j$ and simply by interchanging $x_j$ and $x_{j-1}$ we could obtain a term of higher order. Thus our result follows directly from Theorem 2.

HOFSTRA COLLEGE

# NON-INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A LINEAR CONGRUENCE

M. L. VEST

1. **Introduction.** Vogt[1] has studied, using synthetic methods, the space transformations associated with a linear congruence and with a bundle of lines. In the present paper the non-involutorial transformations associated with these configurations are found analytically, and by an entirely different geometric process. It will be shown that the general transformation associated with a linear congruence so obtained differs from that of Vogt in one important respect, having one more fundamental curve in each of the projectively related spaces. The transformations in the planes containing the directrices of the congruence are also shown.

Given the directrices $r$ and $s$ of the congruence and two projective pencils of surfaces

$$|F_{m+n+1}| : r^m s^n g_{2m+2n+2mn+1}$$

and

$$|F'_{m'+n'+1}| : r^{m'} s^{n'} g_{2m'+2n'+2m'n'+1}.$$

Through a generic point $P(y)$ there passes a single $F$ of $|F|$. The unique line $t$ of the congruence through $P(y)$ meets the associated $F'$ of $|F'|$ in one residual point $P'(x)$, the image of $P(y)$ under the transformation thus defined. The residual base curves of $|F|$ and $|F'|$, other than $r$ and $s$, have been denoted by $g$ and $g'$, respectively. Through a point $O_{g'}$ on $g'$ there is a unique line $t'$ of the congruence, this line lying upon one surface of $|F'|$. The associated surface of $|F|$ meets $t'$ in a point $\overline{P}$ which generates a curve $\bar{g}$. Similarly, beginning with a point $O_g$ on $g$, a point $\overline{P}'$ generating a curve $\bar{g}'$ is found. It will be shown that $r$, $s$, $g$, $g'$, $\bar{g}$, $\bar{g}'$ are fundamental curves of the transformation.

2. **Equations of the transformation.** Let us take the equations of the directrices $r$ and $s$, respectively, as

$$x_1 = x_2 = 0, \qquad x_3 = x_4 = 0, \tag{1}$$

and the pencils of surfaces

$$|F_{m+n+1}| \equiv U - \lambda \overline{U} = 0, \quad |F'_{m'+n'+1}| \equiv U' - \lambda \overline{U}' = 0 \tag{2}$$

Received by the editors January 24, 1942.

[1] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlanddische Kultur, class 84, 1906, pp. 8–16.

where

$$U_{m+n+1} = \sum_{j,k=0}^{m,n} u_j v_k (c_{jk}x), \qquad \overline{U}_{m+n+1} = \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k (\bar{c}_{jk}x),$$

$$U'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} u'_j v'_k (c'_{jk}x), \qquad \overline{U}'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k (\bar{c}'_{jk}x),$$

$$u_j = \sum_{i=0}^{m} a_{ij} x_1^i x_2^{m-i}, \qquad v_k = \sum_{i=0}^{n} b_{ik} x_3^i x_4^{n-i},$$

(3)

$$\bar{u}_j = \sum_{i=0}^{m} \bar{a}_{ij} x_1^i x_2^{m-i}, \qquad \bar{v}_k = \sum_{i=0}^{n} \bar{b}_{ik} x_3^i x_4^{n-i},$$

$$u'_j = \sum_{i=0}^{m'} a'_{ij} x_1^i x_2^{m'-i}, \qquad v'_k = \sum_{i=0}^{n'} b'_{ik} x_3^i x_4^{n'-i},$$

$$\bar{u}'_j = \sum_{i=0}^{m'} \bar{a}'_{ij} x_1^i x_2^{m'-i}, \qquad \bar{v}'_k = \sum_{i=0}^{n'} \bar{b}'_{ik} x_3^i x_4^{n'-i},$$

$$(c_{jk}x) = c_{jk,1}x_1 + c_{jk,2}x_2 + c_{jk,3}x_3 + c_{jk,4}x_4,$$

and so on.

Through a generic point $P(y)$ there is an $F$ of $|F|$ with parameter $\lambda = U(y)/\overline{U}(y)$ and to this corresponds the $F'$ having equation

$$U'(x)\overline{U}(y) - \overline{U}'(x)U(y) = 0. \tag{4}$$

The unique line $t$ of the congruence passing through $P(y)$ intersects the surface (4) in the point $P'$ having coordinates

$$T^{-1}_{m+m'+n+n'+3}: \quad \begin{cases} \sigma x_1 = Ry_1 + Ky_1 = Sy_1 \\ \sigma x_2 = Ry_2 + Ky_2 = Sy_2 \\ \sigma x_3 = Ry_3 \\ \sigma x_4 = Ry_4 \end{cases} \tag{5}$$

where

$$K_{m+m'+n+n'+2} = U'\,\overline{U} - U\,\overline{U}', \quad R_{m+m'+n+n'+2} = U\overline{W}' - \overline{U}W',$$

$$W'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} u'_j v'_k (c'_{jk}z), \qquad \overline{W}'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k (\bar{c}'_{jk}z), \tag{6}$$

$$(c_{jk}z) = c_{jk,1}y_1 + c_{jk,2}y_2.$$

Equations (5) are those of the inverse transformation.

In a similar manner the equations of the direct transformation are found to be

$$(7)\qquad T_{m+m'+n+n'+3}:\ \begin{cases} \sigma y_1 = R'x_1 + K'x_1 = S'x_1 \\ \sigma y_2 = R'x_2 + K'x_2 = S'x_2 \\ \sigma y_3 = R'x_3 \\ \sigma y_4 = R'x_4 \end{cases}$$

where

$$(8)\qquad \begin{gathered} K'_{m+m'+n+n'+2} = U\overline{U}' - \overline{U}U' = -K,\ R'_{m+m'+n+n'+2} = U'\overline{W} - \overline{U}'W, \\ W_{m+n+1} = \sum_{j,k=0}^{m,n} u_j v_k(c_{jk}z), \qquad \overline{W}_{m+n+1} = \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k(\bar{c}_{jk}z). \end{gathered}$$

3. **Images of fundamental curves.** The transformations $T^{-1}$ and $T$ applied to an $F'$ and $F$ of $|F'|$ and $|F|$, respectively, give

$$(9)\qquad U' \sim (T^{-1})US^{m'}R^{n'}G, \qquad U \sim (T)U'S'^{m}R'^{n}G'$$

where

$$(10)\qquad \begin{gathered} G_{2m'+2n'+2} = U'\overline{W}' - \overline{U}'W', \qquad G'_{2m+2n+2} = U\overline{W} - \overline{U}W, \\ S_{m+m'+n+n'+2} = U\overline{M}' - \overline{U}M', \qquad S'_{m+m'+n+n'+2} = U'\overline{M} - \overline{U}'M, \\ M'_{m'+n'+1} = W' - U' = \sum_{j,k=0}^{m',n'} u'_j v'_k(c'_{jk}w), \\ M_{m+n+1} = W - U = \sum_{j,k=0}^{m,n} u_j v_k(c_{jk}w), \\ \overline{M}'_{m'+n'+1} = \overline{W}' - \overline{U}' = \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k(\bar{c}'_{jk}w), \\ \overline{M}_{m+n+1} = \overline{W} - \overline{U} = \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k(\bar{c}_{jk}w), \\ (c_{jk}w) = c_{jk,3}x_3 + c_{jk,4}x_4. \end{gathered}$$

Here $U$ and $U'$ are the corresponding surfaces of $|F|$ and $|F'|$ and $g' \sim (T^{-1})G$, $s' \sim (T^{-1})S$, $g \sim (T)G'$, $s \sim (T)S'$. Similarly,

$$(11)\qquad \begin{gathered} K' \sim (T^{-1})K'R^{n+n'}S^{m+m'}GG', \qquad K \sim (T)KR'^{n+n'}S'^{m+m'}GG', \\ K' \sim (T)K'R'^{n+n'}S'^{m+m'}GG', \qquad K \sim (T)KR^{n+n'}S^{m+m'}GG', \\ G' \sim (T^{-1})S^{2m+1}R^{2n+1}G', \qquad G \sim (T)S'^{2m'+1}R'^{2n'+1}G, \\ G' \sim (T)S'^{2m+1}R'^{2n+1}G', \qquad G \sim (T^{-1})S^{2m'+1}R^{2n'+1}G, \\ R' \sim (T^{-1})S^{m'+m+1}R^{n'+n}GG', \qquad R \sim (T)S'^{m+m'+1}R'^{n+n'}GG', \\ R' \sim (T)S'^{m+m'+1}R'^{n+n'}[R'^2 - K'(W'\overline{W} - \overline{W}'W)], \\ R \sim (T^{-1})S^{m+m'+1}R^{n+n'}[R^2 - K(W'\overline{W} - \overline{W}'W)], \\ S' \sim (T^{-1})S^{m+m'}R^{n+n'+1}GG', \qquad S \sim (T)S'^{m+m'}R'^{n+n'+1}GG', \\ S' \sim (T)S'^{m+m'}R'^{n+n'+1}[R'S' + K'(W'\overline{M} - \overline{W}'M)], \\ S \sim (T^{-1})S^{m+m'}R^{n+n'+1}[RS + K(W\overline{M}' - \overline{W}M')]. \end{gathered}$$

Through a point $O_r$ on $r$ there is a pencil of transversals through $s$. Each direction in this pencil determines an $F'$ and the associated $F$ cuts the line in one residual point. The locus of such points is a plane curve $c$ which generates the surface $R$, image of $r$. The order of $c$, determined by the intersection of $R$ and a homaloidal surface, is $n+n'+1$. Similarly, taking a point $O_s$ on $s$, we find a plane curve $k$ of order $m+m'+1$ generating the surface $S$, image of $s$.

Through a point $O_{g'}$ on $g'$ there is a unique line $t$ of the congruence, but every $F$ of $|F|$ is associated with $O_{g'}$, hence $O_{g'} \sim (T^{-1})t$. The ruled surface $G$ generated by $t$ is the image of $g'$ under $T^{-1}$. Furthermore, every point $P'$ of the transversal determines the same $F'$ and $t$ meets the associated $F$ in one point $\overline{P}$ so that $\overline{P} \sim (T)t$. The locus of points $\overline{P}$ is the curve $\bar{g}$ and $\bar{g} \sim (T)G$. The order of $\bar{g}$, determined by the intersection of two homaloidal surfaces, is $2m'n'+2mn'+2m'n+m+3m'+n+3n'+3$. In a similar manner we find a curve $\bar{g}'$ of order $2mn+2mn'+2m'n+3m+m'+3n+n'+3$ such that $\bar{g}' \sim (T^{-1})G'$.

We can now write the following correspondences:

$$
\begin{aligned}
&r \sim (T)R' \colon\ s^{m+m'+1}r^{n+n'}g'\bar{g}', && r \sim (T^{-1})R \colon\ s^{m+m'+1}r^{n+n'}g\bar{g},\\
&s \sim (T)S' \colon\ s^{m+m'}r^{n+n'+1}g'\bar{g}', && s \sim (T^{-1})S \colon\ s^{m+m'}r^{n+n'+1}g\bar{g},\\
&g \sim (T)G' \colon\ s^{2m+1}r^{2n+1}g\bar{g}', && g' \sim (T^{-1})G \colon\ s^{2m'+1}r^{2n'+1}\bar{g}g',\\
&\bar{g}' \sim (T^{-1})G' \colon\ s^{2m+1}r^{2n+1}g\bar{g}', && \bar{g} \sim (T)G \colon\ s^{2m'+1}r^{2n'+1}\bar{g}g'.
\end{aligned}
\tag{12}
$$

4. **Invariant and homaloidal surfaces.** The eliminant of the paramter from $|F|$ and $|F'|$ is the pointwise invariant surface $K$. Generic planes subjected to the transformations give

$$
\begin{aligned}
\pi' \equiv (A'x) &\sim (T^{-1})R(A'y) + K(A'z) = \phi_{m+m'+n+n'+3},\\
\pi \equiv (Ay) &\sim (T)R'(Ax) + K'(Az) = \phi'_{m+m'+n+n'+3}
\end{aligned}
$$

where the $\phi$'s are homaloidal surfaces of the transformation. Further,

$$
\begin{aligned}
\phi &\sim (T)(A'x)S'^{m+m'+1}R'^{n+n'+1}GG',\\
\phi' &\sim (T^{-1})(Ay)S^{m+m'+1}R^{n+n'+1}GG',
\end{aligned}
$$

hence the homaloidal webs are

$$
\infty^3 |\phi| \colon\ s^{m+m'+1}r^{n+n'+1}g\bar{g}, \qquad \infty^3 |\phi'| \colon\ s^{m+m'+1}r^{n+n'+1}g'\bar{g}'.
$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$
\begin{aligned}
H &\equiv [\phi\phi] \colon s^{m^2+m'^2+2mm'+2m+2m'+1}r^{n^2+2nn'+n'^2+2n+2n'+1}g\bar{g}b_{m+m'+n+n'+3},\\
H' &\equiv [\phi'\phi'] \colon s^{m^2+2mm'+m'^2+2m+2m'+1}r^{n^2+2nn'+n'^2+2n+2n'+1}g'\bar{g}'b'_{m+m'+n+n'+3}.
\end{aligned}
$$

We now write the additional correspondences:

$$(13)\quad \begin{aligned} &\pi \sim (T)\phi' \colon\ s^{m+m'+1}r^{n+n'+1}g'\bar{g}', \quad \pi \sim (T^{-1})\phi \colon\ s^{m+m'+1}r^{n+n'+1}g\bar{g}, \\ &K \sim (T)K' \colon\ s^{m+m'}r^{n+n'}gg'\bar{g}\bar{g}', \quad K' \sim (T^{-1})K \colon\ s^{m+m'}r^{n+n'}gg'\bar{g}\bar{g}'. \end{aligned}$$

The jacobian of the transformation is $J=RGG'S$.

5. **Tangency along $r$ and $s$.** The projectivity $x_1=y_1$, $x_2=y_2$, $x_3=ky_3$, $x_4=y_3+y_4$ is applied to the fundamental surfaces of the transformation and an examination of the coefficients of the highest powers of $x_3$ shows $K$ and $S$ to have

$$\begin{aligned} D_{m+m'} \equiv &\sum_{j,k=0}^{m',n'} u_j'\Big(\sum_{i=0}^{n'} b_{ik}k^i\Big)(kc'_{jk,3}+c'_{jk,4}) \\ &\cdot \sum_{j,k=0}^{m,n} \bar{u}_j\Big(\sum_{i=0}^{n} b_{ik}k^i\Big)(k\bar{c}'_{jk,3}+\bar{c}_{jk,4}) \\ &- \sum_{j,k=0}^{m,n} u_j\Big(\sum_{i=0}^{n} b_{ik}k^i\Big)(kc_{jk,3}+c_{jk,4}) \\ &\cdot \sum_{j,k=0}^{m',n'} \bar{u}_j'\Big(\sum_{i=0}^{p'} b'_{ik}k^i\Big)(k\bar{c}'_{jk,3}+\bar{c}'_{jk,4}) = 0 \end{aligned}$$

as common tangent surface along $r$. $D$ is also a common tangent surface to $K'$ and $S'$ along $r$. A similar process shows

$$\begin{aligned} E_{n+n'} \equiv &\sum_{j,k=0}^{m',n'} v_k'\Big(\sum_{i=0}^{m'} a'_{ij}k^i\Big)(k\bar{c}'_{jk,1}+\bar{c}'_{jk,2}) \\ &\cdot \sum_{j,k=0}^{m,n} \bar{v}_k\Big(\sum_{i=0}^{m} \bar{a}_{ij}k^i\Big)(k\bar{c}_{jk,1}+\bar{c}_{jk,2}) \\ &- \sum_{j,k=0}^{m,n} v_k\Big(\sum_{i=0}^{m} a_{ij}k^i\Big)(kc_{jk,1}+c_{jk,2}) \\ &\cdot \sum_{j,k=0}^{m',n'} \bar{v}_k'\Big(\sum_{i=0}^{m'} \bar{a}'_{ij}k^i\Big)(k\bar{c}'_{jk,1}+\bar{c}'_{jk,2}) = 0 \end{aligned}$$

to be common tangent surface to $K$, $K'$, $R$ and $R'$ along $s$.

6. **Intersection table.** Referring to (12), (13), and §5 we can now write the following intersection table.

$$\begin{aligned} &[R'S']\colon\ s^{m^2+m'^2+2mm'+m+m'}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}' \\ &[R'G']\colon\ s^{2m^2+2mm'+3m+m'+1}r^{2n^2+2nn'+n'+n}\bar{g}'d_{1,1}\cdots d_{1,2mn+2n} \\ &[R'\phi']\colon\ s^{m^2+m'^2+2mm'+2m+2m'+1}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'c_{n+n'+1} \\ &[R'K']\colon\ s^{m^2+m'^2+2mm'+m+m'}r^{n'^2+2nn'+n^2+(n+n')t}g'\bar{g}' \end{aligned}$$

$$[S'G']\colon\ s^{2m^2+2mm'+m+m'}\gamma^{2n'+2nn^2+3n+n'+1}\bar{g}'e_{1,1}\cdots e_{1,2mn+2m}$$

$$[S'\phi']\colon\ s^{m^2+m'^2+2mm'+m+m'}\gamma^{n^2+n'^2+2nn'+2n+2n'+1}g'\bar{g}'k_{m+m'+1}$$

$$[S'K']\colon\ s^{m^2+2mm'+m'^2+(m+m')}t_\gamma{}^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'$$

$$[G'\phi']\colon\ s^{2m^2+2m'm+3m+m'+1}\gamma^{2n^2+2nn'+3n+n'+1}\bar{g}'l_{1,1}\cdots l_{1,2m+2mn+2n+1}$$

$$[G'K']\colon\ s^{2m^2+m+2mm'+m'}\gamma^{2n^2+n+2nn'+n'}g\bar{g}'$$

$$[\phi'K']\colon\ s^{m^2+m'^2+2mm'+m+m'}\gamma^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'c_{m+m'+n+n'+2}.$$

**7. The $T_{n+n'+2}$ in a plane through $r$.** A plane $\pi\equiv x_1=\delta x_2$ cuts the surfaces of $|F|$ and $|F'|$ in residual pencils of curves

$$|f_{n+1}|\equiv u-\mu\bar{u}=0,\quad |f'_{n'+1}|\equiv u'-\mu\bar{u}'=0 \tag{14}$$

where

$$u_{n+1}=\sum_{j,k=0}^{m,n}\psi_j\omega_k(\alpha_{jk}x),\qquad \bar{u}_{n+1}=\sum_{j,k=0}^{m,n}\bar{\psi}_j\bar{\omega}_k(\bar{\alpha}_{jk}x),$$

$$u'_{n'+1}=\sum_{j,k=0}^{m',n'}\psi'_j\omega'_k(\alpha'_{jk}x),\qquad \bar{u}'_{n'+1}=\sum_{j,k=0}^{m',n'}\bar{\psi}'_j\bar{\omega}'_k(\bar{\alpha}'_{jk}x),$$

$$\psi_j=\sum_{i=0}^{m}a_{ij}\delta^i,\quad \bar{\psi}_j=\sum_{i=0}^{m}\bar{a}_{ij}\delta^i,\quad \psi'_j=\sum_{i=0}^{m'}a'_{ij}\delta^i,\quad \bar{\psi}'_j=\sum_{i=0}^{m'}\bar{a}'_{ij}\delta^i, \tag{15}$$

$$\omega_k=\sum_{i=0}^{n}b_{ik}x_3{}^ix_4{}^{n-i},\qquad \bar{\omega}_k=\sum_{i=0}^{n}\bar{b}_{ik}x_3{}^ix_4{}^{n-i},$$

$$\omega'_k=\sum_{i=0}^{n'}b'_{ik}x_3{}^ix_4{}^{n'-i},\qquad \bar{\omega}'_k=\sum_{i=0}^{n'}\bar{b}'_{ik}x_3{}^ix_4{}^{n'-i},$$

$$(\alpha_{jk}x)=(\alpha_{jk,1}\delta+\alpha_{jk,2})x_2+\alpha_{jk,3}x_3+\alpha_{jk,4}x_4.$$

The second directrix $s$ cuts $\pi$ in the point $P$: (δ, 1, 0, 0) which is an $n$-fold point on the curves of $|f|$ and an $n'$-fold point on the curves of $|f'|$. Through a generic point $P(y)$ of $\pi$ passes an $f$ of $|f|$ having parameter $\mu=u(y)/\bar{u}(y)$ and to this corresponds the $f'$ of $|f'|$ having equation

$$u'(x)\bar{u}(y)-\bar{u}'(x)u(y)=0. \tag{16}$$

The unique transversal $t$ through $P(y)$ and $P$ meets (16) in a point $P'(x)$, image of $P(y)$ under the transformation. The inverse transformation is thus

$$T^{-1}_{n+n'+2}\colon\ x_2=\sigma y_2+\kappa,\qquad x_3=\sigma y_3,\qquad x_4=\sigma y_4, \tag{17}$$

where

$$\kappa_{n+n'+2} = u'\bar{u} - \bar{u}'u \qquad \sigma_{n+n'+1} = u\bar{w}' - \bar{u}w',$$

$$(18) \qquad w'_{n'} = \sum_{j,k=0}^{m',n'} \psi'_j \omega'_k(\alpha'_{jk}z), \qquad \bar{w}'_{n'} = \sum_{j,k=0}^{m',n'} \bar{\psi}'_j \bar{\omega}'_k(\bar{\alpha}'_{jk}z),$$

$$(\alpha_{jk}z) = \alpha_{jk,1}\delta + \alpha_{jk,2}$$

while the direct transformation is

$$(19) \qquad T_{n+n'+2}\colon\ y_2 = \sigma' x_2 + \kappa', \qquad y_3 = \sigma' x_3, \qquad y_4 = \sigma' x_4$$

where

$$\kappa' = -\kappa, \qquad \sigma'_{n+n'+1} = u'\bar{w} - \bar{u}'w,$$

$$(20) \qquad w_n = \sum_{j,k=0}^{m,n} \psi_j \omega_k(\alpha_{jk}z), \qquad \bar{w}_n = \sum_{j,k=0}^{m,n} \bar{\psi}_j \bar{\omega}_k(\bar{\alpha}_{jk}z).$$

The curve $\kappa$ is pointwise invariant under the transformations. The application of the transformation and its inverse gives

$$(21) \qquad u'(x) \sim (T^{-1})\sigma^{n'}u\gamma, \qquad u(y) \sim (T)\sigma'^{n}u'\gamma',$$

where $\gamma_{2n'+1} = u'\bar{w}' - \bar{u}'w'$, $\gamma'_{2n+1} = u\bar{w} - \bar{u}w$, and, furthermore,

$$\kappa' \sim (T^{-1})\kappa'\sigma^{n+n'}\gamma\gamma', \qquad \kappa \sim (T)\kappa\sigma'^{n+n'}\gamma\gamma',$$

$$\kappa' \sim (T)\kappa'\sigma'^{n+n'}\gamma\gamma', \qquad \kappa \sim (T^{-1})\kappa\sigma^{n+n'}\gamma\gamma',$$

$$\gamma' \sim (T^{-1})\sigma^{2n+1}\gamma', \qquad \gamma \sim (T)\sigma'^{2n'+1}\gamma,$$

$$\gamma' \sim (T)\sigma'^{2n+1}\gamma', \qquad \gamma \sim (T^{-1})\sigma^{2n'+1}\gamma,$$

$$\sigma' \sim (T^{-1})\sigma^{n+n'}\gamma\gamma', \qquad \sigma \sim (T)\sigma'^{n+n'}\gamma\gamma',$$

$$\sigma' \sim (T)\sigma'^{n+n'}\left[\sigma'^{2} + \kappa'(w'\bar{w} - w\bar{w}')\right],$$

$$\sigma \sim (T^{-1})\sigma^{n+n'}\left[\sigma^{2} + \kappa(w'\bar{w} - w\bar{w}')\right].$$

The base points of the pencil $|f|$ are $P$ and $2n+1$ other points $\Gamma_1, \Gamma_2, \cdots, \Gamma_{2n+1}$ while those of $|f'|$ are $P$ and $\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_{2n'+1}$. We designate the lines $P\Gamma_1, P\Gamma_2, \cdots, P\Gamma_{2n+1}$ as $\gamma'_1, \gamma'_2, \cdots, \gamma'_{2n+1}$ and $P\Gamma'_1, P\Gamma'_2, \cdots, P\Gamma'_{2n'+1}$ as $\gamma_1, \gamma_2, \cdots, \gamma_{2n'+1}$. The curves $\gamma$ and $\gamma'$ in (21) consist of the lines $\gamma_1, \gamma_2, \cdots, \gamma_{2n'+1}$ and $\gamma'_1, \gamma'_2, \cdots, \gamma'_{2n+1}$, respectively. It is easily seen that $\Gamma_i \sim (T)\gamma'_i$, $\Gamma'_i \sim (T^{-1})\gamma_i$, $\gamma'_i \sim (T^{-1})\Gamma_i$, $\gamma_i \sim (T)\Gamma'_i$.

Moreover, there is a particular $f$ of $|f|$ consisting of the lines $\gamma'_1$ and an $n$-ic through $\Gamma_2, \Gamma_3, \cdots, \Gamma_{2n+1}, P$, the latter being an $(n-1)$-fold point on the $n$-ic. The $n$-ic intersects $\gamma'_1$ in another point $\bar{P}_1$. Then $\bar{P}_1 \sim (T)\bar{\Gamma}'_1$ and $\bar{\Gamma}'_1 \sim (T^{-1})\gamma'_1$. Similarly, we can find points $\Gamma'_2, \cdots, \Gamma'_{2n+1}, \Gamma_1, \cdots, \Gamma_{2n'+1}$ such that $\bar{\Gamma}'_i \sim (T^{-1})\gamma'_i$ and $\bar{\Gamma}_i \sim (T)\gamma_i$.

We can now write the following correspondences:

$$P \sim (T)\rho' \colon\ P^{n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}',$$

$$\Gamma_i \sim (T)\gamma_i' \colon\ P\Gamma_i\bar{\Gamma}_i',$$

$$\bar{\Gamma}_i' \sim (T^{-1})\gamma_i' \colon\ P\Gamma_i\bar{\Gamma}_i',$$

$$\kappa \sim (T)\kappa' \colon\ P^{n+n'}\Gamma_1 \cdots \Gamma_{2n+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1 \cdots \bar{\Gamma}_{2n'+1}\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

When the transformations are applied to a generic line we get

$$\lambda' \equiv (Ax) \sim (T^{-1})\sigma(Ay) + \kappa A_2 = \phi_{n+n'+2},$$

$$\lambda \equiv (Ay) \sim (T)\sigma'(Ax) + \kappa' A_2 = \phi'_{n+n'+2},$$

$$\phi \sim (T)(Ay)\sigma'^{n+n'+1}\gamma\gamma', \qquad \phi' \sim (T^{-1})(Ax)\sigma^{n+n'+1}\gamma\gamma'$$

and the holaloidal nets are therefore

$$\infty^2 \left| \phi \right| \colon\ P^{n+n'+1}\Gamma_1 \cdots \Gamma_{2n+1}\bar{\Gamma}_1 \cdots \bar{\Gamma}_{2n'+1},$$

$$\infty^2 \left| \phi' \right| \colon\ P^{n+n'+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

The translation $x_2 = x_2' + 1$, $x_3 = x_3'$, $x_4 = x_4'$ is applied to the fundamental curves of the transformation and $\sigma'$ and $\kappa'$ are found to have the curve

$$l_{n+n'} \equiv \sum_{j,k=0}^{m',n'} \psi'\omega'(\alpha'_{jk,1}\delta + \alpha'_{jk,2}) \cdot \sum_{j,k=0}^{m,n} \bar{\psi}\bar{\omega}(\bar{\alpha}_{jk,1}\delta + \bar{\alpha}_{jk,2})$$
$$- \sum_{j,k=0}^{m',n'} \bar{\psi}'\bar{\omega}'(\bar{\alpha}'_{jk,1}\delta + \bar{\alpha}'_{jk,2}) \cdot \sum_{j,k=0}^{m,n} \psi\omega(\alpha_{jk,1}\delta + \alpha_{ik,2}) = 0$$

as common tangent at the point $P$.

The jacobian is $\sigma^2\gamma\gamma'$.

The intersection table is as follows

$$[\phi'\phi'] \colon\ P^{n^2+2nn'+n'^2+2n+2n'+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'F$$

(the homaloidal web)

$$[\sigma'\gamma_i'] \colon\ P^{n+n'}\bar{\Gamma}_i', \qquad [\gamma_i'\kappa'] \colon\ P^{n+n'}\Gamma_i\bar{\Gamma}_i', \qquad [\gamma_i'\phi'] \colon\ P^{n+n'+1}\bar{\Gamma}_i'$$

$$[\sigma'\kappa'] \colon\ P^{n^2+2nn'+n'^2+(n+n')^t}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'$$

$$[\sigma'\phi'] \colon\ P^{n^2+2nn'+n'^2+n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'$$

$$[\kappa'\phi'] \colon\ P^{n^2+2nn'+n'^2+n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

As the plane $\pi$ generates the pencil on $r$ the $T_{n+n'+2}$ generates the

space $T_{m+m'+n+n'+2}$ whose equations may be obtained from (17) and (19) by replacing $u$, $\bar{u}$, $u'$, $\bar{u}'$, $w$, $\bar{w}$, $w'$, $\bar{w}'$, $(\alpha z)$ and $\delta$ by $U$, $\overline{U}$, $U'$, $\overline{U}'$, $W$, $\overline{W}$, $W'$, $\overline{W}'$, $(cz)$ and $x_1/x_3$, respectively.

When the surfaces are cut by a plane $\pi' \equiv x_3 = \epsilon x_4$ through $s$ a similar plane transformation of order $m+m'+2$ is found.

8. **The transformation associated with a special linear congruence.** If we take the directrices $r$ and $s$ of the congruence as

$$x_1 = x_4 = 0, \qquad x_2 = x_4 = 0, \tag{22}$$

respectively, and pencils of surfaces $F$ and $F'$ as in (2) and (3) where $u_j = \sum_{i=0}^{m} a_{ij} x_1^i x_4^{m-i}$, $v_k = \sum_{i=0}^{n} b_{ik} x_2^i x_4^{n-i}$ and similarly for $\bar{u}_j$, $u_j'$, and so on, the equations of the inverse transformation obtained by the present technique are

$$T^{-1}_{m+m'+n+n'+2}\colon\ x_1 = Ry_1,\ \ x_2 = Ry_2,\ \ x_3 = Ry_3 + K,\ \ x_4 = Ry_4 \tag{23}$$

where $K_{m+m'+n+n'+2} = U'\overline{U} - U\overline{U}'$, $R_{m+m'+n+n'+1} = U\overline{W}' - \overline{U}W'$,

$$W'_{m'+n'} = \sum_{j,k=0}^{m',n'} u_j' v_k' (c'_{jk,3}), \qquad \overline{W}'_{m'+n'} = \sum_{j,k=0}^{m',n'} \bar{u}_j' \bar{v}_k' (\bar{c}'_{jk,3}).$$

Such transformations are well known so that no discussion will be given.

9. **Discussion.** Vogt, in his general skew transformation, finds three fundamental curves and their images in each space. These he designates as $g_1$, $g_2$, $l_1$, $l_2$, $R_1$, $R_2$ having images $\gamma_2$, $\gamma_1$, $\lambda_2$, $\lambda_1$, $\rho$, $\rho$, respectively. Of these $g_1$, $g_2$, $l_1$, $l_2$, $\gamma_2$, $\gamma_1$, $\lambda_2$, $\lambda_1$, correspond to $r$, $r$, $s$, $s$, $R'$, $R$, $S'$, $S$ of the present paper. However, $R_1$, $R_2$, $\rho$, $\rho$, are replaced by $g$, $\bar{g}$; $g'$, $\bar{g}'$; $G'$, $G$; $G$, $G'$, respectively. The sums of the orders of $g$ and $\bar{g}$, $g'$ and $\bar{g}'$, $G$ and $G'$ are equal to the orders of $R_1$, $R_2$, and $\rho$, respectively, as given by Vogt.

The properties of the transformation in §8 are the same as those of the "central" transformation which he describes.

Quadratic transformations associated with a linear congruence may of course be gotten from those of the present paper by taking $|F|$ and $|F'|$ as pencils of planes, that is, making $m=n=0$, in the plane transformations of §7 and the space transformation of §8.

WEST VIRGINIA UNIVERSITY

# THE MEASURE OF THE CRITICAL VALUES OF DIFFERENTIABLE MAPS

ARTHUR SARD

**1. Introduction.** Consider the *map*

$$y^j = f^j(x^1, x^2, \cdots, x^m), \qquad j = 1, 2, \cdots, n, \tag{1.1}$$

of a region $R$ of euclidean $m$-space into part of euclidean $n$-space. Suppose that each function $f^j$ $(j=1, \cdots, n)$ is of class $C^q$ in $R$ $(q \geqq 1)$.[1] A *critical point* of the map (1.1) is a point in $R$ at which the matrix of first derivatives $\mathfrak{M} = \|f_i^j\|$ $(i=1, \cdots, m; j=1, \cdots, n)$ is of less than maximum rank. The *rank* of a critical point $x$ is the rank of $\mathfrak{M}$ at $x$. A *critical value* is the image under (1.1) of a critical point. If $n=1$, these definitions are the usual definitions of critical point and value of a continuously differentiable function.

We prove the following result: *If $m \leqq n$, the set of critical values of the map* (1.1) *is of $m$-dimensional measure*[2] *zero* without further hypothesis on $q$; *if $m > n$, the set of critical values of the map* (1.1) *is of $n$-dimensional measure zero providing that $q \geqq m-n+1$.* Using an example due to Hassler Whitney[3] we show that the hypothesis on $q$ cannot be weakened. We prove also that the critical values of (1.1) corresponding to critical points of rank zero constitute a set of $(m/q)$-dimensional measure zero.

The idea of considering the measure of the set of critical values of one function or of several functions is due to Marston Morse.

The first result stated above reduces, if $n=1$, to the known theorem: The critical values of a function of $m$ variables of class $C^m$ constitute a set of linear measure zero. A. P. Morse[4] has given a proof of this theorem for all $m$. In the present paper we make use of one of A. P. Morse's results.

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Presented to the Society, January 1, 1941 under the title *The measure of the critical values of differentiable maps of euclidean spaces*; received by the editors February 9, 1942.

[1] A function is of class $C^q$ if all its partial derivatives of order $q$ exist and are continuous.

[2] In the sense of Hausdorff-Saks. The definition is given in §2.

[3] H. Whitney, *A function not constant on a connected set of critical points*, Duke Mathematical Journal, vol. 1 (1935), pp. 514–517.

[4] A. P. Morse, *The behaviour of a function on its critical set*, Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70. Proofs for the cases $m=1, 2, 3$ had previously been given by M. Morse and for the cases $m=4, 5, 6$ by M. Morse and the author in unpublished papers.

In a study of functional dependence,[5] A. B. Brown has shown that the set of critical values of (1.1) corresponding to a closed bounded set of critical points is nowhere dense providing that $q$ satisfies certain conditions. The theorems of the present paper imply this particular result.

2. **$s$-dimensional measure.** Given a set $A$ and positive quantities $s$ and $\alpha$. Denote the diameter of $A$ by $\delta(A)$. Let $\{A_1, A_2, \cdots\}$ be a covering of $A$ by sets of diameter less than $\alpha$; and let $L_s(A;\alpha)$ be the greatest lower bound of the sums $\sum_k[\delta(A_k)]^s$ for all such coverings. The *$s$-dimensional outer measure* of $A$ is

$$(2.1) \qquad L_s(A) = c_s \lim_{\alpha \to 0} L_s(A;\alpha), \qquad c_s = \pi^{s/2}/2^s\Gamma[(s+2)/2],$$

where $c_s$ is, for integral $s$, the $s$-dimensional volume of a sphere of unit diameter in $s$-space.[6]

We shall say that $A$ is an *$s$-null* set if $L_s(A)=0$. The value of $c_s$ in (2.1) is immaterial to this paper as we are concerned with the nullity of sets. An $s$-null set is a fortiori an $(s+p)$-null set ($p$ positive). $L_s(A)$ is a regular Carathéodory measure. In $n$-space $L_n(A)=|A|$, where $|A|$ is the outer Lebesgue measure of $A$.[7]

3. **Critical points of positive rank: change of variables.** Consider a critical point $x_0$ of rank $r>0$. Suppose (without real loss of generality) that the determinant $\Delta=|f_c^d|$ $(c, d=1, \cdots, r)$ is not zero at $x_0$. Consider the change of variables from $x$ to $u$ defined by the equations

$$(3.1) \qquad u^d = f^d(x), \quad u^{r+g} = x^{r+g}, \quad d = 1, \cdots, r;\ g = 1, \cdots, m-r.$$

Let $u_0$ be the image of $x_0$ under (3.1); and let $J$ be the Jacobian matrix of (3.1). Since $|J|=\Delta\neq 0$ at $x_0$, the inverse of (3.1)

$$x^i = \phi^i(u), \qquad i = 1, \cdots, m,$$

exists and is of class $C^q$ near $u_0$. In terms of the new variables $u$ the map (1.1) is

$$(3.2) \qquad y^d = u^d = F^d(u), \qquad d = 1, \cdots, r,$$

$$(3.3) \qquad y^{r+h} = f^{r+h}[\phi(u)] = F^{r+h}(u), \qquad h = 1, \cdots, n-r.$$

Let $\mathfrak{M}'$ be the functional matrix $\|F_i^j\|$ $(i=1, \cdots, m;\ j=1, \cdots, n)$.

[5] A. B. Brown, *Functional dependence*, Transactions of this Society, vol. 38 (1935), pp. 379–394.

[6] F. Hausdorff, *Dimension und äusseres Mass*, Mathematische Annalen, vol. 79 (1919), pp. 157–179. S. Saks, *Theory of the Integral*, Warsaw, 1937, pp. 53–54.

[7] We make use of this fact for the case $|A|=0$ only.

Then $\mathfrak{M}=\mathfrak{M}'J$ and therefore $\mathfrak{M}$ and $\mathfrak{M}'$ have the same rank since $|J|\neq 0$. Hence *we may consider* (3.2, 3.3) *near* $u_0$ *instead of* (1.1) *near* $x_0$ *without changing either critical values or ranks of critical points.*

4. **Case I:** $m\leqq n$. We prove Theorem 4.1.

THEOREM 4.1. *The critical values of the map* (1.1) *constitute an* $m$*-null set if* $m\leqq n$.

PROOF. We consider first the set $A$ of critical points of rank zero. We shall show that there is a neighborhood $N$ of each point of $A$ such that $f(NA)$, the image of $NA$ under (1.1), is an $m$-null set. As $A$ can be covered by denumerably many such neighborhoods $N$ it will follow that $f(A)$ is an $m$-null set.

Let $N$ be an open $m$-cube[8] whose closure is in $R$. Suppose that $x_1$ is a point of $NA$ and $x_2$ is a point of $N$. Let $y_e=f(x_e)$ $(e=1, 2)$. Then by the mean value theorem,

$$y_2^j - y_1^j = \sum_{i=1}^{m} \xi_i^j (x_2^i - x_1^i), \qquad j = 1, \cdots, n, \tag{4.2}$$

where $\xi_i^j \to 0$ as $x_2 \to x_1$.

Let $C(\gamma)$ be a closed $m$-cube of side $\gamma$. It follows from (4.2) applied twice and the triangle inequality that if $C(\gamma)$ contains a point of $A$, then $\delta\{f[NC(\gamma)]\}\leqq 2m^2\zeta\gamma$ and $\delta\{f[NC(\gamma)]\}^m \leqq (2m^2\zeta)^m |C(\gamma)|$, where $\zeta$ is the least upper bound of the functions $|\xi_i^j|$ for all $x_1$ in $NA$, $x_2$ in $N$ such that $|x_2^i - x_1^i|\leqq\gamma$ $(i=1, \cdots, m;\ j=1, \cdots, n)$. Then $\zeta\to 0$ with $\gamma$.

Therefore given $\alpha>0$ and $\epsilon>0$, there exists a $G>0$ such that if $C(\gamma)$ contains a point of $A$, $\gamma<G$ implies that

$$\delta\{f[NC(\gamma)]\} < \alpha, \qquad \delta\{f[NC(\gamma)]\}^m < \frac{\epsilon}{|N|+1}|C(\gamma)|. \tag{4.3}$$

Consider the set of all cubes $C(\gamma)$ centered at points of $NA$ and such that $\gamma<G$. By a theorem related to Vitali's covering theorem,[9] there exists a sequence $\{C_1, C_2, \cdots\}$ of cubes of this set which covers all of $NA$ and is such that

[8] By a cube we mean a cube with sides parallel to the axes.

[9] Hans Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik, vol. 27 (1916), Theorem II, p. 190, with "circles centered at $P$" replaced by "squares centered at $P$." It is not necessary to use the theorem related to Vitali's covering theorem. One may instead consider a network of cubes; or one may give a proof like A. P. Morse's proof of his Theorem 4.3 (loc. cit., pp. 68–69). This remark applies also to our later applications of the theorem related to Vitali's covering theorem. The author is indebted to A. P. Morse for his having suggested the applicability of Vitali's covering theorem.

$$\sum_k |C_k| \leqq |N| + 1. \tag{4.4}$$

The sequence $\{f(NC_k)\}$ covers $f(NA)$. Let $\delta_k$ be the diameter of $f(NC_k)$. Then (4.3) and (4.4) imply that $\delta_k < \alpha$ $(k=1, 2, \cdots)$ and

$$\sum_k \delta_k^m < \frac{\epsilon}{|N|+1} \sum_k |C_k| \leqq \epsilon.$$

Hence $f(NA)$ is an $m$-null set. This completes the first part of the proof.

We consider now the set $B$ of critical points of rank $r$, where $r$ is any positive integer less than $m$. To prove that $f(B)$ is an $m$-null set we shall show that there is a neighborhood $N$ of each point of $B$ such that $f(NB)$ is an $m$-null set.

Consider a point $x_0$ of $B$. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed cube $K$ centered at $u_0$. Let $D$ be the set of critical points of (3.2, 3.3) of rank $r$ in $K$.

Suppose that $u_1$ is a point of $D$ and $u_2$ is a point of $K$. Let $y_e = F(u_e)$ $(e=1, 2)$. Then by the mean value theorem,

$$\begin{aligned} y_2^d - y_1^d &= u_2^d - u_1^d, && d = 1, \cdots, r, \\ y_2^{r+h} - y_1^{r+h} &= \sum_{c=1}^{r} F_c^{r+h}(u_1)(u_2^c - u_1^c) + \sum_{i=1}^{m} \zeta_i^{r+h}(u_2^i - u_1^i), && h = 1, \cdots, n-r, \end{aligned} \tag{4.5}$$

where $\zeta_i^{r+h} \to 0$ as $u_2 \to u_1$.

Let $C(\gamma)$ be a closed cube of side $\gamma$ in the space $(u^1, \cdots, u^m)$; and let $\Pi(\gamma)$ be the projection of $C(\gamma)$ on the space $(u^1, \cdots, u^r)$. Let $p$ be a positive integer. Divide $\Pi(\gamma)$ by bisections into $2^{rp}$ congruent closed $r$-cubes each of side $\gamma/2^p$. Each such subcube is the projection of a *strip* of $C(\gamma)$ in which $u^1, \cdots, u^r$ may change by at most $\gamma/2^p$ and $u^{r+1}, \cdots, u^m$ may change by at most $\gamma$.

Consider any strip $S$ that contains a point of $D$. Let $u_1$ and $u_2$ be any two points of $S$ and let $y_e = F(u_e)$ $(e=1, 2)$. Then (4.5) applied twice implies that

$$\begin{aligned} |y_2^d - y_1^d| &\leqq \gamma/2^p, && d = 1, \cdots, r, \\ |y_2^{r+h} - y_1^{r+h}| &\leqq 2rU\gamma/2^p + 2m\zeta\gamma, && h = 1, \cdots, n-r, \end{aligned} \tag{4.6}$$

where $U$ is the least upper bound of the functions $|F_c^{r+h}(u)|$ for all $u$ in $K$ and $\zeta$ is the least upper bound of the functions $|\zeta_i^{r+h}|$ for all $u_1$ in $D$, $u_2$ in $K$ such that $|u_2^i - u_1^i| \leqq \gamma$ $(c=1, \cdots, r;\ h=1, \cdots, n-r;$ $i=1, \cdots, m)$. Then $\zeta \to 0$ with $\gamma$.

Let $\delta$ be the diameter of $F(S)$. The triangle inequality and (4.6) imply that

$$\delta \leqq (V/2^p + W\zeta)\gamma, \tag{4.7}$$

where $V=r+(n-r)2rU$ and $W=(n-r)2m$. $V$ and $W$ are constants, and $V \geqq r \geqq 1$.

Let $\alpha$ and $\epsilon$ be given, $\alpha>0$, $1>\epsilon>0$. Put

$$\eta = \epsilon/(|K| + 1). \tag{4.8}$$

There exists a number $G$,

$$0 < G < \alpha/(V + 1), \tag{4.9}$$

such that $\gamma<G$ implies that

$$W\zeta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}/2 < 1. \tag{4.10}$$

Let $p$ be the positive integer determined by the relation

$$2V(4V)^{r/(m-r)}\eta^{-1/(m-r)} < 2^p \leqq 4V(4V)^{r/(m-r)}\eta^{-1/(m-r)}. \tag{4.11}$$

(Note that the first member of (4.11) is greater than 1, as it should be.)

Then $\gamma<G$ implies that $\delta<\alpha$ by (4.7), (4.9) and (4.10), since $2^p>1$; and $\gamma<G$ also implies that

$$\delta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}\gamma \tag{4.12}$$

by (4.7), the first half of (4.11) and (4.10). Now consider all strips $S$ containing points of $D$. There are at most $2^{rp}$ such strips; for these

$$\sum \delta^m < 2^{rp}\eta^{m/(m-r)}(4V)^{-mr/(m-r)}\gamma^m \leqq \eta\gamma^m = \eta\,|C(\gamma)|, \tag{4.13}$$

by (4.12) and the second half of (4.11). Thus $\gamma<G$ implies that there are sets covering $F[DC(\gamma)]$ each of whose diameters is less than $\alpha$ and the sum of the $m$th powers of whose diameters is less than $\eta|C(\gamma)|$.

Consider the set of all cubes $C(\gamma)$ centered at points of $D$ and such that $\gamma<G$. As before, a sequence $\{C_k\}$ of these cubes covers $D$ and is such that

$$\sum_k |C_k| \leqq |K| + 1. \tag{4.14}$$

Then the covering of $F(D)$ consisting of all the coverings of the sets $F(DC_k)$ is a covering by sets each of whose diameters is less than $\alpha$ and the sum of the $m$th powers of whose diameters is less than $\eta\sum_k|C_k| \leqq \epsilon$, by (4.13), (4.8) and (4.14). Hence $F(D)$ is an $m$-null set.

Let $N$ be the inverse image under (3.1) of the interior of $K$. Then $N$

is a neighborhood of $x_0$. Further $f(NB)$ is contained in $F(D)$ and is therefore an $m$-null set. This completes the proof of Theorem 4.1.

5. **A. P. Morse's theorem.** We state the following theorem, due to A. P. Morse.[10]

THEOREM 5.1. *Given a positive integer $q$ and a set $A$ in the space of the variables $x$. There exists a sequence $A_0, A_1, A_2, \cdots$ of sets with the following properties*: (i) $A=\sum_{k=0}^{\infty} A_k$, (ii) $A_0$ *is denumerable*, (iii) $A_k$ $(k=1, 2, \cdots)$ *is bounded*, (iv) *if $g(x)$ is any function of class $C^q$ whose critical set includes $A$ and if $x_1$ and $x_2$ are points of $A_k$, then*

$$\lim_{x_2 \to x_1} \frac{g(x_2) - g(x_1)}{|x_2 - x_1|^q} = 0, \qquad k = 1, 2, \cdots.$$

6. **Critical points of rank zero.** We prove Theorem 6.1.

THEOREM 6.1. *Let $A$ be the set of critical points of rank zero of the map* (1.1). *Then $f(A)$ is an $s$-null set if $s \geqq m/q$.*

PROOF. Decompose $A$ into the subsets of Theorem 6.1. Then $f(A_0)$ is denumerable and hence is an $s$-null set. We shall prove that $f(A_k)$ is an $s$-null set $(k=1, 2, \cdots)$. It will follow that $f(A)$ is an $s$-null set.

Consider a non-empty $A_k$ ($k$ fixed and positive). For simplicity put $B=A_k$. Given $\alpha>0$ and $\epsilon>0$. Given a point $x$ of $B$. Let $C(\gamma)$ be the closed cube of side $\gamma$ *centered at* $x$. By continuity and Theorem 5.1 there is a positive $G_x<1$ such that $\gamma<G_x$ implies that

$$\delta\{f[C(\gamma)]\} < \alpha \tag{6.2}$$

and

$$|f^j(x_1) - f^j(x)| < \frac{\epsilon^{1/s} 2^{q-1}}{m^q n(|B|+1)^{1/s}} |x_1 - x|^q, \quad j = 1, \cdots, n, \tag{6.3}$$

whenever $x_1$ is in $BC(\gamma)$. Now by the triangle inequality $|x_1-x| \leqq m\gamma/2$ if $x_1$ is in $BC(\gamma)$. Hence (6.3) twice and the triangle inequality imply that

$$\delta\{f[BC(\gamma)]\} < 2 \frac{\epsilon^{1/s} 2^{q-1}}{m^q(|B|+1)^{1/s}} \left(\frac{m\gamma}{2}\right)^q = \frac{\epsilon^{1/s}/\gamma^q}{(|B|+1)^{1/s}},$$

and therefore

$$\delta\{f[BC(\gamma)]\}^s < \epsilon\gamma^{qs}/(|B|+1), \tag{6.4}$$

providing that $\gamma<G_x$.

[10] Loc. cit., Theorem 4.2. By denumerable we mean denumerably infinite or finite. The sets $A_k$ $(k=1, 2, \cdots)$ are condensed; however we do not make use of this property. The symbol $|x_2-x_1|$ denotes the distance between $x_1$ and $x_2$.

Consider the set of all cubes $C(\gamma)$, $\gamma < G_x$, centered at points $x$ of $B$. As before, a sequence $\{C_k\}$ of these cubes covers $B$ and is such that

$$\sum_k |C_k| \leqq |B| + 1. \tag{6.5}$$

Let $\gamma_k$ be the side of $C_k$ $(k=1, 2, \cdots)$.

The sequence $\{f(BC_k)\}$ covers $f(B)$. Further $\delta[f(BC_k)] < \alpha$ by (6.2); and

$$\sum_k \delta[f(BC_k)]^s < \epsilon \sum_k \gamma_k^{qs}/(|B| + 1) \leqq \epsilon \sum_k |C_k|/(|B| + 1) \leqq \epsilon$$

by (6.4) and (6.5), since $\gamma_k \leqq 1$ by construction, $qs \geqq m$ by hypothesis and $|C_k| = \gamma_k^m$ $(k=1, 2, \cdots)$. Hence $f(B)$ is an $s$-null set.

7. **Case II: $m > n$.** We prove Theorems 7.1 and 7.2.

THEOREM 7.1. *Let $A$ be the set of critical points of rank $r$ of the map* (1.1). *Then $f(A)$ is an $n$-null set if $q \geqq (m-r)/(n-r)$, $m > n$.*

PROOF. If $r=0$ the theorem reduces to Theorem 6.1 with $s=n$.

Suppose that $0 < r < n$. To prove that $f(A)$ is an $n$-null set we shall show that there is a neighborhood $N$ of each point of $A$ such that $f(NA)$ is an $n$-null set.

Consider a point $x_0$ of $A$. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed neighborhood $\overline{N}$ of $u_0$. Regard $u^1, \cdots, u^r$ as parameters for each permissible set of values of which (3.3) defines a map of the $(m-r)$-space $(u^{r+1}, \cdots, u^m)$ into the $(n-r)$-space $(y^{r+1}, \cdots, y^n)$. For each $(u^1, \cdots, u^r)$ let $\mathfrak{M}^*$ be the functional matrix of (3.3). Then $\mathfrak{M}'$ is of rank $r$ if and only if $\mathfrak{M}^*$ is of rank zero.

Thus if $(u^1, \cdots, u^m)$ is a critical point of (3.2, 3.3) of rank $r$, then $(u^{r+1}, \cdots, u^m)$ is a critical point of (3.3) of rank zero for the values $u^1, \cdots, u^r$ of the parameters. But for each set of values of the parameters, the critical points of (3.3) of rank zero map into an $(n-r)$-null set, by Theorem 6.1 and hypothesis.

Let $B$ be the set of critical points of (3.2, 3.3) of rank $r$ in $\overline{N}$. The cross-section of $F(B)$ for each $(y^1, \cdots, y^r)$ is thus an $(n-r)$-null set. Further $F(B)$ is closed and therefore measurable. Hence $F(B)$ is an $n$-null set by the theorem of Fubini.

THEOREM 7.2. *If $m > n$ the critical values of the map* (1.1) *constitute an $n$-null set providing that $q \geqq m-n+1$.*

PROOF. Apply Theorem 7.1 with $r=0, 1, \cdots, n-1$. For these values of $r$, $m-n+1 \geqq (m-r)/(n-r)$.

**8. The hypothesis on $q$ in Theorem 7.2 cannot be weakened.** Let $W(x^1, \cdots, x^t)$ be a function of $t$ variables, $x^1, \cdots, x^t$, of class $C^{t-1}$, which takes on every value from 0 to 1 on a Jordan arc of critical points ($t \geqq 2$). H. Whitney has constructed such a function.[11] Consider the map

$$y^1 = W(x^1, \cdots, x^{m-n+1}), \qquad y^a = x^{m-n+a}, \tag{8.1}$$
$$a = 2, 3, \cdots, n; m > n,[12]$$

in the unit cube. The map (8.1) is of class $C^{m-n}$, but the critical values of (8.1) do not constitute an $n$-null set. Indeed the set of critical values of (8.1) is the unit cube and is thus of $n$-dimensional measure 1.

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[11] Loc. cit.

[12] If $n=1$, (8.1) is to consist of the equation for $y^1$ only.

# THE RADICAL OF A NON-ASSOCIATIVE ALGEBRA

A. A. ALBERT

**1. Introduction.** An algebra $\mathfrak{A}$ is said to be nilpotent of index $r$ if every product of $r$ quantities of $\mathfrak{A}$ is zero, and is said to be a zero algebra if it is nilpotent of index two. It is said to be simple if it is not a zero algebra and its only nonzero (two-sided) ideal is itself, and is said to be semi-simple if it is a direct sum of simple algebras.

The radical of an associative algebra $\mathfrak{A}$ is a nilpotent ideal $\mathfrak{N}$ of $\mathfrak{A}$ which is maximal in the strong sense in that it contains[1] all nilpotent ideals of $\mathfrak{A}$. No such ideal exists in an arbitrary non-associative algebra, and so the radical of such an algebra has never[2] been defined. However the property that $\mathfrak{A}-\mathfrak{N}$ be semi-simple is really the vital one and we shall define the concept of radical here by proving this theorem.

THEOREM 1. *Every algebra $\mathfrak{A}$ which is homomorphic to a semi-simple algebra has an ideal $\mathfrak{N}$, which we shall call the* **radical** *of $\mathfrak{A}$, such that $\mathfrak{A}-\mathfrak{N}$ is semi-simple, $\mathfrak{N}$ is contained in every ideal $\mathfrak{B}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{B}$ is semi-simple.*

The hypothesis that $\mathfrak{A}$ shall be homomorphic to a semi-simple algebra is equivalent to the property that there shall exist an ideal $\mathfrak{B}$ in $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ shall be semi-simple. It is a necessary assumption even in the associative case, since $\mathfrak{A}$ may be nilpotent and then $\mathfrak{A}=\mathfrak{N}$, every $\mathfrak{A}-\mathfrak{B}$ is nilpotent. Moreover it is satisfied by every algebra $\mathfrak{A}$ with a unity quantity. We shall, nevertheless, carry our study a step farther in that we shall define explicitly a certain proper ideal $\mathfrak{N}$ for every algebra $\mathfrak{A}$ such that either $\mathfrak{N}$ is the radical of $\mathfrak{A}$ in the sense above or $\mathfrak{A}$ is not homomorphic to a semi-simple algebra. In the latter case $\mathfrak{A}-\mathfrak{N}$ is a zero algebra.

Our results will be consequences of the remarkable fact[3] that the major structural properties of any non-associative algebra $\mathfrak{A}$ over $\mathfrak{F}$ are determined by almost the same properties of a certain related associative algebra $T(\mathfrak{A})$. We define the right multiplications $R_x$ and the

Presented to the Society April 18, 1942; received by the editors February 11, 1942.

[1] For these results see my *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, chap. 2.

[2] For the case of alternative algebras see M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Annals of Mathematics, (2), vol. 42 (1941), pp. 676–686.

[3] Cf. my *Non-associative algebras* I: *Fundamental concepts and isotopy*, Annals of Mathematics, (2), vol. 43 (1942), pp. 685–708. See also N. Jacobson, *A note on non-associative algebras*, Duke Mathematical Journal, vol. 3 (1937), pp. 544–548.

left multiplications $L_x$ of $\mathfrak{A}$ for every $x$ of $\mathfrak{A}$ to be the respective linear transformations

$$a \to a \cdot x = aR_x, \qquad a \to x \cdot a = aL_x, \qquad a \text{ in } \mathfrak{A},$$

on $\mathfrak{A}$, and let the transformation algebra $T(\mathfrak{A})$ of $\mathfrak{A}$ be the polynomial ring over $\mathfrak{F}$ generated by the $R_x$, the $L_x$, and the identity transformation $I$. If $\mathfrak{S}$ is any set of linear transformations $S$ on $\mathfrak{A}$ we define $\mathfrak{A}\mathfrak{S}$ to be the linear subspace of $\mathfrak{A}$ spanned by the images $aS$ of every $a$ of $\mathfrak{A}$. Then if $\mathfrak{H}$ is the radical of $T(\mathfrak{A})$ the set $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$ which is zero if and only if $\mathfrak{H}=0$. When $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra the algebra $T(\mathfrak{A})-\mathfrak{H}$ is a field of order one and we shall prove these theorems.

THEOREM 2. *An algebra $\mathfrak{A}$ is homomorphic to a semi-simple algebra if and only if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is not a zero algebra.*

THEOREM 3. *If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is a zero algebra if and only if $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$.*

THEOREM 4. *Let $\mathfrak{A}$ be homomorphic to a semi-simple algebra. Then either $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple and $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$ or $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is the direct sum of a semi-simple algebra and a zero algebra $\mathfrak{N}_0=\mathfrak{N}-\mathfrak{A}\mathfrak{H}$ such that $\mathfrak{N}$ is the radical of $\mathfrak{A}$.*

We shall close our discussion with a study of algebras with a unity quantity and the radicals of isotopes with unity quantities. Moreover we shall exhibit an algebra with a unity quantity and a radical which is a field.

2. **A fundamental lemma.** Let $\mathfrak{B}$ be a linear subspace of an algebra $\mathfrak{A}$ of order $n$ over $\mathfrak{F}$ and $m$ be the order of $\mathfrak{B}$ so that there exists an idempotent $E$ of rank $m$ in the algebra $(\mathfrak{F})_n$ of all linear transformations on $\mathfrak{A}$ such that

$$\mathfrak{B} = \mathfrak{A}E.$$

Then $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ if and only if

$$ET(\mathfrak{A}) = ET(\mathfrak{A})E. \tag{1}$$

Since $T(\mathfrak{A})$ contains the identity transformation $I$ it follows that

$$\mathfrak{B}T(\mathfrak{A}) = \mathfrak{B}. \tag{2}$$

We let $\mathfrak{S}$ be the intersection

$$T(\mathfrak{A})E \cap T(\mathfrak{A}), \tag{3}$$

so that $\mathfrak{S}$ consists of all $S=SE$ in $T(\mathfrak{A})$. Then $\mathfrak{S}T(\mathfrak{A})$ is contained in $T(\mathfrak{A})ET(\mathfrak{A})=T(\mathfrak{A})ET(\mathfrak{A})E$ by (1), $\mathfrak{S}T(\mathfrak{A})$ is contained in $\mathfrak{S}$. Also $T(\mathfrak{A})T(\mathfrak{A})E=T(\mathfrak{A})E$, $T(\mathfrak{A})\mathfrak{S}=\mathfrak{S}$ is an ideal[4] of $T(\mathfrak{A})$. Then $T(\mathfrak{A})-\mathfrak{S}$ is defined and we may prove the fundamental lemma.

LEMMA 1. *The algebra* $T(\mathfrak{A}-\mathfrak{B})$ *is equivalent to* $T(\mathfrak{A})-\mathfrak{S}$.

For let $a$ be any quantity of $\mathfrak{A}$ and $\{a\}=a+\mathfrak{B}$ be the corresponding coset in the decomposition of the additive group $\mathfrak{A}$ relative to $\mathfrak{B}$. If $T$ is in $T(\mathfrak{A})$ the set $\mathfrak{B}T$ is contained in $\mathfrak{B}$, the coset $\{aT\}=aT+\mathfrak{B}$ is independent of $a$. Then the correspondence

$$a \to \{aT\} = \{a\}T_0 \tag{4}$$

is a transformation $T_0$ of $\mathfrak{A}-\mathfrak{B}$ uniquely determined for every $T$ of $T(\mathfrak{A})$. Moreover $T_0$ is a linear transformation. But then we have determined a mapping

$$T \to T_0 \tag{5}$$

of $T(\mathfrak{A})$ on a set $\mathfrak{T}_0$ of linear transformations $T_0$ on $\mathfrak{A}-\mathfrak{B}$. It is clear from (4) that

$$\{a(T_1\alpha + T_2\beta)\} = \{aT_1\}\alpha + \{aT_2\}\beta = \{a\}(T_{10}\alpha + T_{20}\beta), \tag{6}$$

$$\{a(T_1T_2)\} = \{(aT_1)T_2\} = \{aT_1\}T_{20} = \{a\}T_{10}T_{20} \tag{7}$$

for every $\alpha$ and $\beta$ of $\mathfrak{F}$, $T_1$ and $T_2$ of $T(\mathfrak{A})$. Then (5) determines a homomorphism of $T(\mathfrak{A})$ on $\mathfrak{T}_0$.

The general right multiplication $R_{\{x\}}$ of $\mathfrak{A}-\mathfrak{B}$ is the transformation $\{a\}\to\{a\}\cdot\{x\}$, and this is the transformation $(R_x)_0$ given by (5). For

$$\{a\}\cdot\{x\} = \{a\cdot x\} = \{aR_x\}. \tag{8}$$

Similarly $L_{\{x\}}=(L_x)_0$, $\mathfrak{T}_0$ contains $T(\mathfrak{A}-\mathfrak{B})$. If $u_1, \cdots, u_n$ are a basis of $\mathfrak{A}$ over $\mathfrak{F}$ and $S_i=R_{u_i}$, $T_i=L_{u_i}$ every transformation of $T(\mathfrak{A})$ is a polynomial $T=\phi(I, S_1, \cdots, S_n, T_1, \cdots, T_n)$, $T_0=\phi(I, S_{10}, \cdots, S_{n0}, T_{10}, \cdots, T_{n0})$ is in $T(\mathfrak{A}-\mathfrak{B})$, $T(\mathfrak{A}-\mathfrak{B})=T_0$. If $T_0=0$ we have $\{aT\}=0$ for every $a$, $aT$ is in $\mathfrak{B}$ for every $a$ of $\mathfrak{A}$, $aT=aTE$, $T=TE$ is in $\mathfrak{S}$. Thus the algebra $T(\mathfrak{A})$ is homomorphic under (5) to $T(\mathfrak{A}-\mathfrak{B})$ such that $\mathfrak{S}$ is the ideal of all transformations $T$ of $T(\mathfrak{A})$ such that $T_0=0$. Then $T(\mathfrak{A})-\mathfrak{S}$ is equivalent to $T(\mathfrak{A}-\mathfrak{B})$. This proves our lemma.

3. **Algebras with a semi-simple transformation algebra.** A quantity

[4] This proof is so brief that I repeat it rather than refer to the proof in the article quoted in Footnote 3.

$z \neq 0$ of an algebra $A$ is called an absolute divisor of zero if $z \cdot a = a \cdot z = 0$ for every $a$ of $A$. Suppose first that $\mathfrak{A}$ contains no absolute divisor of zero and that $T(\mathfrak{A})$ is semi-simple. Then $T(\mathfrak{A}) = \mathfrak{T}_1 \oplus \cdots \oplus \mathfrak{T}_r$ for simple algebras $\mathfrak{T}_i$ and $\mathfrak{T}_i = T(\mathfrak{A})E_i$ where $E_i$ is the unity quantity of $\mathfrak{T}_i$, $E_i$ is a nonzero idempotent of the center of $T(\mathfrak{A})$. We let $\mathfrak{A}_i = \mathfrak{A}E_i$ and have (1) for each $E_i$, $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$. Now $\mathfrak{A}_i = \mathfrak{A}_i E_i$ and it follows that the $\mathfrak{A}_i$ are supplementary ideals $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$. Write $\mathfrak{A} = \mathfrak{A}_i \oplus \mathfrak{B}_i$ and have $T(\mathfrak{A}) = \mathfrak{T}_i \oplus \mathfrak{S}_i$, $\mathfrak{T}_i = T(\mathfrak{A})E_i$, $\mathfrak{S}_i = T(\mathfrak{A})E_{i0}$ where $E_{i0} = I - E_i$. Then $\mathfrak{S}_i$ is the algebra $\mathfrak{S}$ of Lemma 1 for $\mathfrak{B} = \mathfrak{B}_i$, $T(\mathfrak{A} - \mathfrak{B}) \simeq T(\mathfrak{A}) - \mathfrak{S}_i$. Clearly $\mathfrak{A} - \mathfrak{B}_i = \mathfrak{A}_i$, $T(\mathfrak{A}) - \mathfrak{S}_i = \mathfrak{T}_i$, $T(\mathfrak{A}_i) = \mathfrak{T}_i$ is simple. But $\mathfrak{A}_i$ has no absolute divisor of zero and then is known[5] to be simple when $\mathfrak{T}_i$ is simple. Thus we have shown that if $T(\mathfrak{A})$ is semi-simple and $\mathfrak{A}$ has no absolute divisors of zero it is semi-simple.

If $\mathfrak{B}$ is a linear subset of $\mathfrak{A}$ we have $\mathfrak{B}\mathfrak{A} \subset \mathfrak{B}T(\mathfrak{A})$, $\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}T(\mathfrak{A})$. Then if $\mathfrak{H}$ is any right ideal of $T(\mathfrak{A})$ we have $(\mathfrak{A}\mathfrak{H})\mathfrak{A} \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$, $\mathfrak{A}(\mathfrak{A}\mathfrak{H}) \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$. Hence $\mathfrak{A}\mathfrak{H}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{H} \neq 0$ is a nilpotent ideal of $T(\mathfrak{A})$ we cannot have $\mathfrak{A}\mathfrak{H} = 0$. Also $\mathfrak{A}\mathfrak{H} \neq \mathfrak{A}$ since otherwise $\mathfrak{H}^t = 0$ would imply that $\mathfrak{A} = 0$. Let then $\mathfrak{A}$ be semi-simple, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$. We write $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ for simple algebras $\mathfrak{A}_i$ and may choose pairwise orthogonal idempotents $E_i$ of $(\mathfrak{F})_n$ such that $\mathfrak{A}_i = \mathfrak{A}E_i$, $E_1 + \cdots + E_r = I$. Then $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$, $E_i T(\mathfrak{A}) = E_i T(\mathfrak{A}) E_i$, $E_i \mathfrak{H} = E_i \mathfrak{H} E_i = \mathfrak{H}_i$ is clearly a nilpotent ideal of $E_i T(\mathfrak{A})$. But it follows that $\mathfrak{A}\mathfrak{H} = \mathfrak{A}_1\mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{A}_1\mathfrak{H}_r$ for ideals $\mathfrak{A}_i\mathfrak{H}_i$ of $\mathfrak{A}_i$. This is impossible unless each $\mathfrak{H}_i = 0$ since each $\mathfrak{A}_i$ is simple. Thus if $\mathfrak{A}$ is semi-simple so is $T(\mathfrak{A})$.

Suppose finally that $\mathfrak{A}$ does have absolute divisors of zero and let $\mathfrak{N}$ be the set of all absolute divisors of zero of $\mathfrak{A}$. Then clearly $\mathfrak{N}$ is an ideal of $\mathfrak{A}$ which is a zero algebra, $\mathfrak{N} = \mathfrak{N}E$ for an idempotent $E$ of $(\mathfrak{F})_n$. If $\mathfrak{A} = \mathfrak{N}$ we have $T(\mathfrak{A}) = I\mathfrak{F}$ and $T(\mathfrak{A})$ is semi-simple. Otherwise $I - E = E_0$ is an idempotent of $(\mathfrak{F})_n$, $\mathfrak{A} = \mathfrak{A}(E + E_0)$ is the direct sum $\mathfrak{A} = \mathfrak{G} \oplus \mathfrak{N}$ where $\mathfrak{G} = \mathfrak{A}E_0$ contains no absolute divisors of zero. If $a$ and $x$ are in $\mathfrak{A}$ we write $x = g + h$ with $g$ in $\mathfrak{G}$ and $h$ in $\mathfrak{N}$, $a \cdot x = a \cdot g = aR_g$, $R_x = R_g$, $(a \cdot x)E_0$ is in $\mathfrak{G}$, $R_x E_0 = R_x$ for every $x$ of $\mathfrak{A}$. Similarly every $L_x$ is in $T(\mathfrak{A})E_0$ and it is clear that $T(\mathfrak{G})$ is equivalent to $T(\mathfrak{A})E_0$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{G}$ is $T(\mathfrak{A})E_0$ and is an ideal of $T(\mathfrak{A})$, $T(\mathfrak{A}) = T(\mathfrak{A})E_0 + I\mathfrak{F}$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{N}$ is $T(\mathfrak{A})E$ and is $EF$ since $E_0 E = 0$. Then $T(\mathfrak{A}) = T(\mathfrak{A})E_0 \oplus EF$, $T(\mathfrak{A})E_0$ is semi-simple when $T(\mathfrak{A})$ is semi-simple, $T(\mathfrak{G})$ is semi-simple

[5] In the paper referred to in Footnote 3, N. Jacobson defined $T(\mathfrak{A})$ to be generated by the right and left multiplications of $\mathfrak{A}$ and with *I omitted*. He then proved our result. We require the more general statement including the case where $\mathfrak{A}$ may be a zero algebra and so refer to Lemma 10 of my own paper of that reference.

and so is $\mathfrak{G}$. Conversely, if $\mathfrak{G}$ is semi-simple so is $T(\mathfrak{A})E_0$ and so is $T(\mathfrak{A})$. We have proved this lemma.

LEMMA 2. *The transformation algebra $T(\mathfrak{A})$ is semi-simple if and only if $\mathfrak{A}$ is either semi-simple, a zero algebra, or a direct sum of a semi-simple algebra and a zero algebra.*

4. **The radical of an algebra.** If $\mathfrak{A}$ is an associative semi-simple algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ we have $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, where $\mathfrak{C}$ is an ideal of $\mathfrak{A}$ equivalent to $\mathfrak{A}-\mathfrak{B}$ and is semi-simple. Let $\mathfrak{A}$ now be an associative algebra with radical $\mathfrak{N}\neq 0$ and $\mathfrak{B}$ be an ideal of $\mathfrak{A}$. If $\mathfrak{B}$ contains $\mathfrak{N}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{N})-(\mathfrak{B}-\mathfrak{N})$ and is semi-simple by the argument above. Conversely, let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ is semi-simple, $\mathfrak{A}-\mathfrak{B}$ contains no properly nilpotent classes. Then every properly nilpotent quantity of $\mathfrak{A}$ must define the zero class of $\mathfrak{A}-\mathfrak{B}$, $\mathfrak{B}$ contains all properly nilpotent quantities of $\mathfrak{A}$, $\mathfrak{B}$ contains $\mathfrak{N}$. This proves Theorem 1 in the associative case.

We now let $\mathfrak{B}$ be any ideal of an arbitrary algebra $\mathfrak{A}$, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$ so that $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple so is $T(\mathfrak{A}-\mathfrak{B})$ by Lemma 2, and so is $T(\mathfrak{A})-\mathfrak{S}$ by Lemma 1. But by the result just proved $\mathfrak{S}$ contains $\mathfrak{H}$. However $\mathfrak{B}=\mathfrak{A}E, \mathfrak{S}=T(\mathfrak{A})E=\mathfrak{S}E$, $\mathfrak{H}=\mathfrak{H}E$, $\mathfrak{A}\mathfrak{H}=\mathfrak{A}\mathfrak{H}E$ is contained in $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ and hence in $\mathfrak{B}$. Then we have proved that the radical $\mathfrak{N}$ of $\mathfrak{A}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$. Hence $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple our definition implies that $\mathfrak{A}\mathfrak{H}=\mathfrak{N}$. Otherwise $\mathfrak{A}_0=\mathfrak{A}-\mathfrak{A}\mathfrak{H}=\mathfrak{G}\oplus\mathfrak{N}_0$ where $\mathfrak{G}$ is semi-simple and $\mathfrak{N}_0$ is a zero algebra, $\mathfrak{B}-\mathfrak{A}\mathfrak{H}$ is an ideal $\mathfrak{B}_0$ of $\mathfrak{A}_0$ such that $\mathfrak{A}_0-\mathfrak{B}_0$ is semi-simple. If there is a quantity of $\mathfrak{N}_0$ not in $\mathfrak{B}_0$ the corresponding class of $\mathfrak{A}_0-\mathfrak{B}_0$ is an absolute divisor of zero of that algebra, contrary to our hypothesis. Hence $\mathfrak{B}_0$ contains $\mathfrak{N}_0$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of all the quantities in the classes of $\mathfrak{N}_0$, $\mathfrak{N}$ is the radical of $\mathfrak{A}$. This proves Theorem 1 and Theorem 4.

If $\mathfrak{A}$ is homomorphic to an algebra $\mathfrak{A}_0$ and $\mathfrak{B}$ is the set of all quantities of $\mathfrak{A}$ mapped into zero by the given homomorphism then $\mathfrak{A}-\mathfrak{B}$ is equivalent to $\mathfrak{A}_0$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple we have seen that $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$, $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. This proves Theorem 2. We have also seen that if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{A}-\mathfrak{B}$ is a zero algebra then $T(\mathfrak{A}-\mathfrak{B})=T(\mathfrak{A})-\mathfrak{S}$ for an ideal $\mathfrak{S}$ of $T(\mathfrak{A})$. But then by Lemma 2 $T(\mathfrak{A})-\mathfrak{S}$ is semi-simple, $\mathfrak{S}$ contains $\mathfrak{H}$, $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ contains $\mathfrak{A}\mathfrak{H}$ and is contained in $\mathfrak{A}E=\mathfrak{B}$. This proves Theorem 3.

5. **The radicals of isotopic algebras.** Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be algebras of the same order so that we may regard them as having quantities in the

same linear space. Then $\mathfrak{A}$ and $\mathfrak{A}_1$ are principal isotopes if multiplication in $\mathfrak{A}_1$ is given by $[a, x]=aR_x^{(1)}$ for $R_x^{(1)}=PR_{xQ}$, $P$ and $Q$ nonsingular linear transformations on $\mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{A}_1$ have unity quantities the transformations $P$ and $Q$ are in $T(\mathfrak{A})$ and $T(\mathfrak{A})=T(\mathfrak{A}_1)$.

Let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$, $\mathfrak{B}=\mathfrak{A}E$, $ET(\mathfrak{A})=ET(\mathfrak{A})E$. Then $EP=EPE$, $EQ=EQE$ and $EPP^{-1}=E=(EP)(EP^{-1})$, $P_1=EP$ is a nonsingular quantity of $ET(\mathfrak{A})$. Similarly $Q_1=EQ$ is a nonsingular quantity of $ET(\mathfrak{A})$. Write $\mathfrak{B}_1=\mathfrak{A}_1E$, so that since $ET(\mathfrak{A}_1)=ET(\mathfrak{A}_1)E$ the space $\mathfrak{B}_1$ is an ideal[6] of $\mathfrak{A}_1$. Then if $b$ and $y$ are in $\mathfrak{B}_1$ we have $b=bE$, $y=yE$,

$$[b, y] = bEPR_{yEQ} = bP_1R_{yQ_1}. \tag{9}$$

It follows that $\mathfrak{B}$ and $\mathfrak{B}_1$ are principal isotopes with isotopy given by

$$R_y^{(1)} = P_1R_{yQ_1}. \tag{10}$$

Every isotope $\mathfrak{A}_1$ of $\mathfrak{A}$ is equivalent to a principal isotope and we have proved the first part of this theorem.

THEOREM 5. *Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be isotopic algebras with unity quantities. Then every ideal $\mathfrak{B}$ of $\mathfrak{A}$ is an isotope of an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$ such that the difference algebras $\mathfrak{A}-\mathfrak{B}$ and $\mathfrak{A}_1-\mathfrak{B}_1$ are isotopes.*

We now observe that the homomorphism (5) of $T(\mathfrak{A})$ on $T(\mathfrak{A}-\mathfrak{B})$ carries every nonsingular $P$ of $T(\mathfrak{A})$ into a nonsingular $P_0$ of $T(\mathfrak{A}-\mathfrak{B})$. Then if we define

$$(R_{\{x\}})^{(1)} = P_0R_{\{x\}Q_0}, \tag{11}$$

the algebra with multiplication defined by

$$[\{a\}, \{x\}] = \{a\}(R_{\{x\}})^{(1)} \tag{12}$$

is a principal isotope of $\mathfrak{A}-\mathfrak{B}$. But the difference algebra $\mathfrak{A}_1-\mathfrak{B}_1$ has multiplication defined by $[\{a\}, \{x\}]=\{[a, x]\}=\{aR_x^{(1)}\}=\{aPR_{xQ}\}$ $=\{aP\}R_{\{xQ\}}=\{a\}(R_{\{x\}})^{(1)}$ since $\{aP\}=\{a\}P_0$, $\{xQ\}=\{x\}Q_0$. This proves our theorem.

We should observe that while $P_0$ and $Q_0$ are in $T(\mathfrak{A}-\mathfrak{B})$ the transformations $P_1$ and $Q_1$ defining the isotopy of $B$ and $B_1$ need not be in $T(\mathfrak{B})$. This is an evident consequence of the fact that if $\mathfrak{A}$ has a unity quantity so does $\mathfrak{A}-\mathfrak{B}$, but certainly $\mathfrak{B}$ need not have a unity quantity. Observe also that if $\mathfrak{A}$ of order $n$ does not have a unity quantity and we pass to an algebra $\underline{\mathfrak{A}}$ of order $n+1$ with a unity quantity the algebra $\mathfrak{A}$ will be an ideal of $\underline{\mathfrak{A}}$. The results above then become of par-

[6] It follows from this that if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the same linear space is an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$. However, we wish to prove the stronger result that $\mathfrak{B}$ and $\mathfrak{B}_1$ are isotopic.

ticular importance in the study of isotopes of algebras *without* unity quantities.

We conclude our general results by proving the following theorem

THEOREM 6. *Let $\mathfrak{A}$ be an algebra with a unity quantity, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$. Then $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$ and is isotopic to the radical $\mathfrak{A}_1\mathfrak{H}_1$ of any isotope $\mathfrak{A}_1$ of $\mathfrak{A}$ with a unity quantity. Moreover the semi-simple algebras $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ and $\mathfrak{A}_1-\mathfrak{A}_1\mathfrak{H}$ are isotopic.*

For every homomorph $\mathfrak{A}-\mathfrak{B}$ of an algebra $\mathfrak{A}$ with a unity quantity has a unity quantity and cannot be a direct sum of a zero algebra and another algebra. Thus Theorem 4 implies that $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$. Our result follows from Theorem 5.

6. **An algebra whose radical is a field.** Let $\mathfrak{A}$ be an algebra with a basis $e, u, v$ over $\mathfrak{F}$ so that every quantity of $\mathfrak{A}$ is uniquely expressible in the form $a=\alpha e+\beta u+\gamma v$ for $\alpha, \beta, \gamma$ in $\mathfrak{F}$. We let $e$ be the unity quantity of $\mathfrak{A}$ and complete the definition of $\mathfrak{A}$ with the relations

$$u^2 = e, \qquad uv = v, \qquad v^2 = v, \qquad vu = 0.$$

Let $\mathfrak{B}$ be a nonzero ideal of $\mathfrak{A}$ and $a\neq 0$ be in $\mathfrak{B}$ so that the corresponding $\alpha, \beta, \gamma$ are not all zero. Then $au=\alpha u+\beta e$, $(au)u=\alpha e+\beta u$, $a-(au)u=\gamma v$, $v[(au)u]=\alpha v$, $v(au)=\beta v$ are all in $\mathfrak{B}$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of order one over $\mathfrak{F}$ spanned by $v$. Now $(\alpha e+\beta u+\gamma v)v=(\alpha+\beta+\gamma)v$, $v(\alpha e+\beta u+\gamma v)=(\alpha+\gamma)v$, $\mathfrak{N}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$ for ideals $\mathfrak{B}$ and $\mathfrak{C}$ we have proved that both $\mathfrak{B}$ and $\mathfrak{C}$ would contain $\mathfrak{N}$. This is impossible. Also $\mathfrak{N}$ is a nonzero proper ideal of $\mathfrak{A}$ and $\mathfrak{A}$ cannot be simple. It follows that $\mathfrak{A}$ is not semi-simple. But $\mathfrak{A}=\mathfrak{N}+\mathfrak{G}$ where $\mathfrak{G}$ is the semi-simple associative algebra spanned by $e$ and $u$, $\mathfrak{A}-\mathfrak{N}=\mathfrak{G}$, $\mathfrak{A}-\mathfrak{N}$ is semi-simple. Then $\mathfrak{N}$ is the radical of $\mathfrak{A}$ according to our definition and is a field of order one over $\mathfrak{F}$.

THE UNIVERSITY OF CHICAGO

# AN ARITHMETICAL IDENTITY FOR THE FORM $ab-c^2$

WALTER H. GAGE

1. **Introduction.** The number of solutions in positive integers of the equation $n=xy+yz+zx$, $n$ a positive integer, has been investigated Liouville,[1] Bell,[2] and Mordell.[3] Mordell, who was the first to obtain complete results, gave a strictly arithmetical treatment, while Bell made use of formulae which he obtained by paraphrasing theta-function identities. Although the latter considered only the case of $n$ prime, his methods were extended later to the general case.[4]

Making use of other formulae derived by the method of paraphrase, Bell[5] has also solved the problem of representations in the forms $xy+yz+2zx$, $xy+2yz+2zx$. As he has pointed out, a feature of the method is the handling of the two forms simultaneously.

In this paper we derive by elementary methods a simple identity which on specialization not only yields complete results for representations of $n$ in the forms

$$xy + yz + zx, \qquad xy + 2yz + 2zx, \qquad xy + yz + 2zx,$$

but as in Bell's paper,[5] handles the latter two forms at the same time.

2. **Fundamental identity.** Let $f(a, b, c)$ be a function, uniform and finite for all integer triples $(a, b, c)$, but otherwise (so far) completely arbitrary. If the summation sign refers to the sum over all those integer solutions $(a, b, c)$ of $n=ab-c^2$ subject to the restrictions indicated under it, we then have

$$\sum_{a,b>c>0} f(a, b, c) = \sum_{a>b>c>0} f(a, b, c) + \sum_{b>a>c>0} f(a, b, c) + \sum_{a=b>c>0} f(a, b, c). \tag{1}$$

Imposing on $f(a, b, c)$ the condition

$$f(a, b, c) = f(b, a, c) \tag{2}$$

Received by the editors February 13, 1942.

[1] Journal de Mathématiques, (2), vol. 7 (1862), p. 44.

[2] E. T. Bell, *Class numbers and the form* $yz+zx+xy$, Tôhoku Mathematical Journal, vol. 19 (1921), pp. 105–116.

[3] L. J. Mordell, *On the number of solutions in positive integers of the equation* $yz+zx+xy=n$, American Journal of Mathematics, vol. 45 (1923), pp. 1–4.

[4] W. H. Gage, *Representations in the form* $xy+yz+zx$, American Journal of Mathematics, vol. 51 (1929), pp. 345–348.

[5] E. T. Bell, *Numbers of representations of integers in a certain triad of ternary quadratic forms*, Transactions of this Society, vol. 32 (1930), pp. 1–5.

we may write

$$\sum_{a,b>c>0} f(a,b,c) = 2\sum_{a>b>c>0} f(a,b,c) + \sum_{a=b>c>0} f(a,b,c)$$

$$= 2\Big[\sum_{a>b>2c>0} f(a,b,c) + \sum_{a>b>c>0,\, b<2c} f(a,b,c) + \sum_{a>b=2c>0} f(a,b,c)\Big] + \sum_{a=b>c>0} f(a,b,c). \tag{3}$$

Note that $n=ab-c^2=a(a+b-2c)-(a-c)^2$, so that if we replace $(a, b, c)$ by $(a+b-2c, a, a-c)$ in the second summation of the right member of (3) the conditions $a>b>c>0$, $b<2c$ become $a, b>2c>0$. Hence, if $f(a, b, c)$ satisfies (2), we have

$$\sum_{a,b>c>0} f(a,b,c) = 2\sum_{a>b>2c>0} f(a,b,c) + 2\sum_{a,b>2c>0} f(a+b-2c, a, a-c)$$

$$+ 2\sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>3\delta>0} f\left(\frac{d+\delta}{2}, 2\delta, \delta\right) \tag{4}$$

$$+ \sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>\delta>0} f\left(\frac{d+\delta}{2}, \frac{d+\delta}{2}, \frac{d-\delta}{2}\right)$$

where the last two summations refer to all integer solutions $(d, \delta)$ of $n=d\delta$ subject to the given conditions.

3. **Specialization.** In (4) let $f(a, b, c)=1$, and in the left member replace $(a, b, c)$ by $(x+z, y+z, z)$. Then, if $N$ denotes the number of integer representations of $n$ in the form stated after it, we get

$$\begin{aligned} N(n &= xy + yz + zx;\ x, y, z > 0) \\ &= 6N(n = ab - c^2;\ a > b > 2c > 0) \\ &\quad + N(n = d\delta;\ d \equiv \delta \pmod 2,\ d > \delta > 0) \\ &\quad + 2N(n = d\delta;\ d \equiv \delta \pmod 2,\ d > 3\delta > 0) \\ &\quad + 2N(n = d\delta;\ d \equiv \delta \pmod 2,\ 3\delta > d > \delta > 0). \end{aligned} \tag{5}$$

Let $G(n)$ denote the number of classes of binary quadratic forms of determinant $-n$, subject to all the conventions of H. J. S. Smith's *Report on the Theory of Numbers.*[6]

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[6] *Mathematical Papers*, vol. 1. See also Dickson's *History of the Theory of Numbers*, vol. 3, pp. 108–109.

$$\begin{aligned} G(n) = {} & N(n = ab - c^2;\ a > b > 2|c| > 0) \\ & + N(n = ab - c^2;\ a > b > 0,\ c = 0) \\ & + N(n = ab - c^2;\ a = b > 2c > 0) \\ & + N(n = ab - c^2;\ a > b = 2c > 0) \\ & + (1/2)N(n = ab - c^2;\ a = b > 0,\ c = 0) \\ & + (1/3)N(n = ab - c^2;\ a = b = 2c > 0). \end{aligned}$$

Hence we have

$$\begin{aligned} & 2N(n = ab - c^2;\ a > b > 2c > 0) \\ & \quad = G(n) - N(n = d\delta;\ d > \delta > 0) - (1/2)N(n = d^2;\ d > 0) \\ & \quad\quad - N(n = d\delta;\ d \equiv \delta \pmod 2,\ d > 3\delta > 0) \\ & \quad\quad - (1/3)N(n = 3d^2;\ d > 0) \\ & \quad\quad - N(n = d\delta;\ d \equiv \delta \pmod 2,\ 3\delta > d > \delta > 0). \end{aligned} \tag{6}$$

From (5) it therefore follows that

$$N(n = xy + yz + zx;\ x, y, z > 0) = 3\left[G(n) - (1/2)\zeta(n)\right], \tag{7}$$

where $\zeta(n)$ is the number of divisors of $n$.

On choosing $f(a, b, c) = 1$ if $a$, $b$ are not both even, $f(a, b, c) = 0$ if $a \equiv b \equiv 0 \pmod 2$, we likewise obtain the result

$$\begin{aligned} & N(n = 2xy + 2yz + 4zx;\ x, y, z > 0,\ y \equiv 1 \pmod 2) \\ & \quad + N(n = xy + 2yz + 2zx;\ x, y, z > 0,\ x \equiv y \equiv 1 \pmod 2) \\ & \quad = F(n) - (1/2)\zeta'(n) - (1/2)\zeta'(n/2), \end{aligned} \tag{8}$$

where $F(n)$ is the number of uneven classes of binary quadratic forms of determinant $-n$, $\zeta'(n)$ is the number of odd divisors of $n$, and $\zeta'(n/2) = 0$ for $n$ odd.

If, in (8), we replace $n$ by $2^{a+1}m$, $2m$, $m$, respectively, $m \equiv 1 \pmod 2$, we get Bell's theorems[5] 1, 2, 3 from which the complete results for $xy+yz+2zx$, $xy+2yz+2zx$ follow.

THE UNIVERSITY OF BRITISH COLUMBIA

# DUAL GEODESICS ON A SURFACE

C. E. SPRINGER

**Introduction.** Union curves and dual union curves have been defined and studied in projective space by Sperry.[1] It is well known that the union curves of the congruence of normals to a metric surface are the geodesics on the surface. The principal aim of this note is to obtain the differential equation of the *dual* geodesics on a metric analytic surface in ordinary space.

The notation of Eisenhart[2] will be employed for the most part. However, $\Gamma^{\alpha}_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will take the range 1, 2, and Latin indices the range 1, 2, 3.

1. **Ray-point corresponding to a point of a curve on the surface.** The tangent planes to the surface $S$ $(x^i = x^i(u^1, u^2))$ at the point $P(x^i)$ and at two "successive" points of the curve $C$ $(u^\alpha = u^\alpha(s))$ on $S$ are given by

$$(\xi^i - x^i)X^i = 0,$$

$$(\xi^i - x^i)\frac{\partial X^i}{\partial u^\alpha}u'^\alpha = 0, \tag{1}$$

$$(\xi^i - x^i)\left(\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta}u'^\alpha u'^\beta + \frac{\partial X^i}{\partial u^\alpha}u''^\alpha\right) = \frac{\partial X^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}u'^\alpha u'^\beta,$$

where the primes indicate differentiation with respect to $s$.

The ray-point[3] $R$ of the curve $C$ corresponding to the point $P$ is the point of intersection of the three planes (1). The coordinates of $R$ are given by

$$S(\xi^i - x^i) = \delta^{ijk}_{\sigma\nu\tau}X^\sigma \frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta}\frac{\partial X^\nu}{\partial u^\gamma}u'^\alpha u'^\beta u'^\gamma, \tag{2}$$

where $i, j, k$ take the values 1, 2, 3 cyclically, $\delta^{ijk}_{\sigma\nu\tau}$ is a Kronecker delta, and $S$ is defined by

Presented to the Society, November 28, 1942; received by the editors of the American Journal of Mathematics February 19, 1942; later transferred to this Bulletin.

[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1928), p. 213.

[2] Eisenhart, *Differential Geometry*, Princeton University Press, 1940.

[3] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932.

$$S \equiv \delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} u'^\gamma \left( \frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta + \frac{\partial X^k}{\partial u^\alpha} u''^\alpha \right). \tag{3}$$

The tangents to the parametric curves at $P$, and the normal to the surface at $P$ constitute a local system of reference. In order to deduce the local coordinates of the point $R$ from equations (2), we employ the formulas from Eisenhart,[4]

$$\frac{\partial X^k}{\partial u^\alpha} = - d_{\alpha\gamma} g^{\gamma\sigma} \frac{\partial x^k}{\partial u^\sigma}, \tag{4}$$

$$\frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} = \frac{\partial X^k}{\partial u^\gamma} \Gamma^\gamma_{\alpha\beta} - h_{\alpha\beta} X^k, \tag{5}$$

$$\Gamma^\mu_{\alpha\beta} = d^{\nu\mu} \frac{\partial d_{\alpha\nu}}{\partial u^\beta} - d^{\nu\mu} d_{\alpha\gamma} \Gamma^\gamma_{\nu\beta}; \tag{6}$$

and we use the fact that

$$\delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^1} \frac{\partial x^k}{\partial u^2} = g^{1/2} \tag{7}$$

to obtain for the first part of the expression for $S$

$$\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} \frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta u'^\gamma = g^{1/2} (g^{\tau 1} g^{\epsilon 2} - g^{\tau 2} g^{\epsilon 1}) d_{\gamma\tau} d_{\mu\epsilon} \Gamma^\mu_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma. \tag{8}$$

On summing out $\tau$ and $\epsilon$, and on writing $|g^{\alpha\beta}| = g^{-1}$, the right member of (8) reduces to

$$g^{-1/2} \delta^{\sigma\tau}_{12} d_{\gamma\sigma} d_{\mu\tau} \Gamma^\mu_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma. \tag{9}$$

In consequence of (4), we have for the second part of the expression for $S$

$$\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\alpha} \frac{\partial X^k}{\partial u^\beta} u'^\alpha u''^\beta = \delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^\sigma} \frac{\partial x^k}{\partial u^\tau} d_{\alpha\gamma} d_{\beta\delta} g^{\gamma\sigma} g^{\delta\tau} u'^\alpha u''^\beta. \tag{10}$$

On summing $\sigma$ and $\tau$, and on making use of (7), the right member of (10) takes the form

$$g^{1/2} (g^{\gamma 1} g^{\delta 2} - g^{\gamma 2} g^{\delta 1}) d_{\alpha\gamma} d_{\beta\delta} u'^\alpha u''^\beta, \tag{11}$$

which, in turn, reduces to

[4] Eisenhart, loc. cit., p. 217, p. 257.

$$g^{-1/2}\delta_{12}^{\sigma\tau}d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}. \tag{12}$$

Because of (9) and (12), equation (3) takes the form

$$S \equiv g^{-1/2}\delta_{12}^{\sigma\tau}(d_{\gamma\sigma}d_{\mu\tau}\Gamma^{\mu}_{\alpha\beta}u'^{\alpha}u'^{\beta}u'^{\gamma} + d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}). \tag{13}$$

In order to express the right member of (2) as a linear combination of the partial derivatives $\partial x^i/\partial u^\alpha$, we use the definition

$$\frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta} = -\,d_{\alpha\beta}, \tag{14}$$

and also the formulas from Eisenhart[5]

$$\begin{aligned}\delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^1} &= eh^{1/2}h^{2\alpha}\frac{\partial X^i}{\partial u^\alpha},\\ \delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^2} &= -\,eh^{1/2}h^{1\alpha}\frac{\partial X^i}{\partial u^\alpha},\end{aligned} \tag{15}$$

where $i, j, k$ take the values 1, 2, 3 cyclically, and where $e$ is $+1$ or $-1$ according as the Gaussian curvature is positive or negative. In consequence of (14), (15), and (4), the right member of (2) assumes the form

$$eh^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}(u'^{1}h^{2\sigma} - u'^{2}h^{1\sigma})d_{\sigma\tau}g^{\tau\delta}\frac{\partial x^i}{\partial u^\delta}. \tag{16}$$

If we multiply the equation[6]

$$h_{\alpha\beta} = d_{\alpha\gamma}d_{\beta\delta}g^{\gamma\delta} \tag{17}$$

by $d^{\alpha\tau}$ and sum on $\alpha$, and then multiply the resulting equation by $h^{\beta\sigma}$ and sum on $\beta$, we obtain

$$d^{\sigma\tau} = h^{\beta\sigma}d_{\beta\delta}g^{\tau\delta}. \tag{18}$$

Because of (18), the expression (16) may be written in the form

$$eh^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}\delta_{\gamma\sigma}^{12}u'^{\gamma}d^{\sigma\delta}\frac{\partial x^i}{\partial u^\delta}, \tag{19}$$

which is the expression desired for the right member of equation (2). Use of (13) and (19) in (2) yields the coordinates $\xi^i$ of the ray-point $R$ of the curve $C$ $(u^\alpha = u^\alpha(s))$ at the point $P(x^i)$ in the form

[5] Eisenhart, loc. cit., p. 259.

[6] Eisenhart, loc. cit., p. 253.

$$(20)\quad \xi^i = x^i + e(hg)^{1/2} d_{\alpha\beta} u'^{\alpha} u''^{\beta} \frac{\delta^{12}_{\gamma\sigma} u'^{\gamma} d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma} d_{\mu\tau} \Gamma^{\mu}_{\alpha\beta} u'^{\alpha} u'^{\beta} u'^{\gamma} + d_{\alpha\sigma} d_{\beta\tau} u'^{\alpha} u''^{\beta})} \cdot \frac{\partial x^i}{\partial u^{\delta}}.$$

The local coordinates $\eta^{\delta}$ of the point $R$ are given by

$$(21)\quad \frac{\eta^{\delta}}{(g_{\delta\delta})^{1/2}} = d d_{\alpha\beta} u'^{\alpha} u'^{\beta} \frac{\delta^{12}_{\gamma\sigma} u'^{\gamma} d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma} d_{\mu\tau} \Gamma^{\mu}_{\alpha\beta} u'^{\alpha} u'^{\beta} u'^{\gamma} + d_{\alpha\sigma} d_{\beta\tau} u'^{\alpha} u''^{\beta})},$$

where $\delta$ is not summed, and where $d \equiv |d_{\alpha\beta}|$ has the value $e(hg)^{1/2}$.

2. **Dual geodesics.** We consider next the line $l_2$ lying in the tangent plane to the surface at $P$ which is in G. M. Green's "relation $R$"[7] to the normal $l_1$ to the surface at $P$. It can readily be shown that the equation of the line $l_2$ in local coordinates is

$$(22)\quad (g_{22})^{1/2} \Gamma^2_{12} \xi^1 + (g_{11})^{1/2} \Gamma^1_{12} \xi^2 + (g_{11} g_{22})^{1/2} = 0.$$

A geodesic curve on the surface has the property that its osculating plane at every point $P$ contains the normal line $l_1$ to the surface at $P$. We may define a *dual geodesic* to be a curve on the surface which enjoys the property that its ray-point $R$ corresponding to every point $P$ lies on the line $l_2$ given by (22). If we require that the coordinates (21) satisfy the equation (22), we find, after some reduction, that the curve $C$ is a solution of the differential equation

$$(23)\quad d\delta^{12}_{\alpha\beta} u'^{\alpha} u''^{\beta} + [d_{\alpha\beta}(d_{\gamma 1}\Gamma^1_{12} - d_{\gamma 2}\Gamma^2_{12}) + \delta^{\sigma\tau}_{12} d_{\gamma\sigma} d_{\mu\tau} \Gamma^{\mu}_{\alpha\beta}] u'^{\alpha} u'^{\beta} u'^{\gamma} = 0,$$

where $\Gamma^{\mu}_{\alpha\beta}$ is given by (6).

If we write $u^1 = u$, $u^2 = v$, $d_{11} = D$, $d_{12} = d_{21} = D'$, $d_{22} = D''$, $v' = dv/du$, $v'' = d^2v/du^2$, and make use of (6), equation (23) takes the form

$$(24)\quad (DD'' - D'^2)v'' = p + qv' + rv'^2 + sv'^3,$$

where

$$\begin{aligned}
p &= D'D_u - DD'_u + (2\Gamma^2_{12} - \Gamma^1_{11})DD' - \Gamma^2_{11} D'^2, \\
q &= D''D_u - DD''_u + D'D_v - DD'_v + (DD'' + D'^2)(2\Gamma^2_{12} - \Gamma^1_{11}) \\
&\quad + D^2\Gamma^1_{22} + DD'(\Gamma^2_{22} - 2\Gamma^1_{12}) - 2D'D''\Gamma^2_{11}, \\
r &= D''D_v - DD''_v + D''D'_u - D'D''_u \\
&\quad + (DD'' + D'^2)(\Gamma^2_{22} - 2\Gamma^1_{12}) - D''^2\Gamma^2_{11} \\
&\quad + D'D''(2\Gamma^2_{12} - \Gamma^1_{11}) + 2DD'\Gamma^1_{22}, \\
s &= D''D'_v - D'D''_v + (\Gamma^2_{22} - 2\Gamma^1_{12})D'D'' + \Gamma^1_{22} D'^2.
\end{aligned} \qquad (25)$$

[7] Lane, loc. cit., p. 81.

If the asymptotic lines are taken as parametric, we have $D=D''=0$, and equation (24) takes the form

$$(26)\qquad v'' = \Gamma_{11}^{2} + (\Gamma_{11}^{1} - 2\Gamma_{12}^{2})v' + (2\Gamma_{12}^{1} - \Gamma_{22}^{2})v'^{2} - \Gamma_{22}^{1}v'^{3}.$$

The geodesics on a surface satisfy the differential equation[8]

$$(27)\qquad v'' = -\Gamma_{11}^{2} + (\Gamma_{11}^{1} - 2\Gamma_{12}^{2})v' + (2\Gamma_{12}^{1} - \Gamma_{22}^{2})v'^{2} + \Gamma_{22}^{1}v'^{3}.$$

From (26) it is seen that if the asymptotic curve given by $v=\text{const.}$ is a dual geodesic, then $\Gamma_{11}^{2}=0$, which, by (27), is the condition that the curve be a geodesic. But if a curve is both an asymptotic and a geodesic it is a straight line. Hence, *if an asymptotic line is a dual geodesic it is a straight line.*

Equation (26) is independent of $D'$. Hence, we conclude that *isometric surfaces have the same equations of dual geodesics.*

It can be shown that when the asymptotics are parametric the directions of Segrè[9] at the point $P$ on the surface are given by

$$(28)\qquad \Gamma_{11}^{2}du^{3} - \Gamma_{22}^{1}dv^{3} = 0.$$

Comparison of equations (26) and (27) shows that the *directions of Segrè may be characterized as the directions in which the geodesics and dual geodesics coincide.*

3. **Ray-points of geodesics.** Each curve through the point $P$ on the surface has a ray-point $R$. We shall now find the locus of the ray-points for all geodesics through $P$. Along a geodesic curve we have

$$(29)\qquad u''^{\alpha} + \Gamma_{\beta\gamma}^{\alpha}u'^{\beta}u'^{\gamma} = 0.$$

In consequence of (29) and the definition of $\overline{\Gamma}_{\alpha\beta}^{\mu}$ from (6), the coordinates $(\eta^1, \eta^2)$ of the ray-point $R$ of a geodesic curve $u^{\alpha}=u^{\alpha}(s)$ through $P$ are found from (21) to be given by

$$(30)\qquad \begin{aligned} T\eta^{1} &= -(g_{11})^{1/2}(u'^{1})^{2}(u'^{2}),\\ T\eta^{2} &= (g_{22})^{1/2}(u'^{1})(u'^{2})^{2}, \end{aligned}$$

where

$$(31)\qquad T = \Gamma_{11}^{2}(u'^{1})^{3} + \Gamma_{12}^{2}(u'^{1})^{2}(u'^{2}) - \Gamma_{12}^{1}(u'^{1})(u'^{2})^{2} - \Gamma_{22}^{1}(u'^{2})^{3}.$$

Elimination of $u'^{1}$ and $u'^{2}$ between the equations (30) yields a cubic equation which may be written in the form

[8] Eisenhart, *Differential Geometry*, Ginn and Co., 1909, p. 205.

[9] Lane, loc. cit., p. 77.

$$\Gamma_{11}^{2}\left(\frac{\eta^{1}}{(g_{11})^{1/2}}\right)^{3}+\Gamma_{22}^{1}\left(\frac{\eta^{2}}{(g_{22})^{1/2}}\right)^{3} - \frac{\eta^{1}\eta^{2}}{(g_{11}g_{22})^{1/2}}\left(\frac{\Gamma_{12}^{2}}{(g_{11})^{1/2}}\eta^{1}+\frac{\Gamma_{12}^{1}}{(g_{22})^{1/2}}\eta^{2}+1\right)=0. \tag{32}$$

From the form of equation (32) it can be seen that the locus has a double point at $P$, with the asymptotics at $P$ as double point tangents. Furthermore, the curve has three points of inflection which lie on the line represented by equation (22), some one of the inflections lying on each of the lines through $P$ given by the equation

$$\Gamma_{11}^{2}\left(\frac{\eta^{1}}{(g_{11})^{1/2}}\right)^{3}+\Gamma_{22}^{1}\left(\frac{\eta^{2}}{(g_{22})^{1/2}}\right)^{3}=0. \tag{33}$$

On comparing equations (28) and (33), we recognize that the latter gives the directions of Darboux[10] at $P$. Hence, we have the following theorem. *The locus of the ray-points of the geodesics through a point P on a surface is a cubic curve lying in the tangent plane to the surface at P. The asymptotics are the double point tangents to the curve at P, and the three points of inflection of the curve are given by the intersections of the tangents of Darboux at P with the line which is in Green's Relation R with the normal to the surface at the point P.* Lane[11] gives a similar theorem for projective space.

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[10] Lane, loc. cit., p. 76.
[11] Lane, loc. cit., p. 98.

# GENERATORS OF PERMUTATION GROUPS SIMPLY ISOMORPHIC WITH $LF(2, p^n)$

F. A. LEWIS

It is well known that the group $LF(2, p^n)$ of linear fractional transformations of determinant unity in the $GF[p^n]$ can be represented as a permutation group $G$ of degree $p^n+1$. The purpose of this note is to show that the generators of $G$ follow from a slight extension of an argument used in a recent paper.[1]

We obtain a representation of the abstract group $L$ simply isomorphic with the special linear homogeneous group $SLH(2, p^n)$ by means of the cosets $K$ and $KTS_\lambda$, where $\lambda$ ranges over the $p^n$ marks of the field $u_0(=0), u_1, \cdots, u_m, (m=p^n-1)$. Let $k_\infty=K$ and $k_{u_i}=KTS_{u_i}$ for $i=0, 1, \cdots, m$.

If $\rho$ is any mark, $KS_\rho=K$ and $KTS_\lambda \cdot S_\rho=KTS_{\lambda+\rho}$, so that to $S_\rho$ there corresponds the permutation

$$(1) \qquad s_\rho = \begin{pmatrix} k_\infty & k_0 & k_{u_1} & \cdots & k_{u_m} \\ k_\infty & k_\rho & k_{u_1+\rho} & \cdots & k_{u_m+\rho} \end{pmatrix}.$$

If $\lambda \neq 0$, $KTS_\lambda T=KTS_{-\lambda^{-1}}$. Further, $KTS_0T=K$, so that to $T$ there corresponds the permutation

$$(2) \qquad t = (k_0 k_\infty \cdot k_{u_1} k_{-u_1^{-1}} \cdots k_{u_m} k_{-u_m^{-1}}).$$

Hence $L$ has a $(d, 1)$ isomorphism with $(s_\rho, t)$, where $d$ is the order of a subgroup of $K$ which is invariant in $L$. The quotient group $(s_\rho, t)$ is simply isomorphic[2] with $LF(2, p^n)$ and is of order $p^n(p^{2n}-1)/d$, where $d=2$ or 1 according as $p>2$ or $p=2$.

THEOREM. *A permutation group simply isomorphic with the group $LF(2, p^n)$ of linear fractional transformations of determinant unity in the $GF[p^n]$ is generated by* (1) *and* (2), *where $\rho$ ranges over an independent set of additive generators of the field.*

COROLLARY.[3] *A permutation group simply isomorphic with the group $LF(2, p)$ is generated by $(k_0k_1k_2 \cdots k_{p-1})$ and $(k_0k_\infty \cdot k_1k_{i_1} \cdot k_2k_{i_2} \cdots)$, where $ji_j \equiv -1 \pmod{p}$.*

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Presented to the Society, December 29, 1939 and April 4, 1942; received by the editors February 20, 1942.

[1] *A note on the special linear homogeneous group $SLH(2, p^n)$*, this Bulletin, vol. 47 (1941), pp. 629–632. The notation and results of this paper are assumed above.

[2] L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, pp. 87–88.

[3] Compare with $x'=x+1$ and $x'=-1/x$, which generate $LF(2, p)$.

# ON AN INEQUALITY OF SEIDEL AND WALSH

LYNN H. LOOMIS

**Introduction.** In a recent paper[1] Seidel and Walsh introduced the following concepts.

Let $R$ be a Riemann surface (configuration) lying over the $w$-plane, and let $C_p$ be a simply-connected region of $R$ having the following properties:

(a) $C_p$ contains precisely $p$ points (counted according to branch-point multiplicity) lying over some point of the $w$-plane.

(b) $C_p$ lies over the circle $|w-w_0|<r$, and the boundary of $C_p$ lies over the circumference $|w-w_0|=r$.

It follows that $C_p$ contains precisely $p$ points lying over every point of $|w-w_0|<r$, and in particular, $p$ points $\bar{w}_i$ lying over $w_0$. Seidel and Walsh name such a region a *$p$-sheeted circle with centers $\bar{w}_i$ and radius $r$*. Given a point $\bar{w}_0$ of $R$, let $r_p$ be the radius of the largest $p$-sheeted circle in $R$ with center $\bar{w}_0$; if none exists, let $r_p=0$. We then define the *radius of $p$-valence of $R$ at $\bar{w}_0$*, $D_p(\bar{w}_0)$, as the maximum of the $r_n$ for $n\leqq p$.

Let $w=f(z)=a_1z+\cdots+a_pz^p+a_{p+1}z^{p+1}+\cdots$ be analytic in the unit circle $|z|<1$ with $|f(z)|<M$, and let the Riemann surface $R$ be the image of $|z|<1$ under $w=f(z)$. Let $\bar{w}_0$ be the image of $z=0$; $\bar{w}_0$ lies over $w=0$. Seidel and Walsh establish the following relation between the first $p$ coefficients of $f(z)$ and the radius of $p$-valence, $D_p(\bar{w}_0)$, of $R$ at $\bar{w}_0$.

*There exist two constants, $\lambda_p$ depending only on $p$, and $\Lambda_p$ depending on $p$ and $M$, such that*

$$\lambda_p D_p(\bar{w}_0) \leqq \sum_{n=1}^{p} |a_n| \leqq \Lambda_p [D_p(\bar{w}_0)]^{2^{-p}}. \tag{1}$$

Seidel and Walsh find for $\Lambda_p$ the value

$$\Lambda_p = 24pM^r, \qquad r = 1-2^{-p}.$$

In this note we prove the following two statements concerning the inequalities (1).

A. *The exponent $2^{-p}$ may be replaced by $1/(p+1)$ and this exponent is the best possible (for $D_p\to 0$).*

Received by the editors February 24, 1942.

[1] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, Transactions of this Society, vol. 52 (1942), pp. 129–216.

We show, in fact, that there is a constant $K_p$ depending only on $p$ such that

$$\sum_{n=1}^{p} |a_n| \leqq K_p M^{p/(p+1)} [D_p(\bar{w}_0)]^{1/(p+1)}, \tag{2}$$

and that there is a class of functions actually attaining the bound $M$ for which

$$\sum_{n=1}^{p} |a_n| \geqq M^{p/(p+1)} [D_p(\bar{w}_0)]^{1/(p+1)}. \tag{3}$$

B. *The coefficient $\lambda_p$ may be replaced by* 1, *and this value of the constant is the best possible.*

All proofs are based on the theorem of Rouché. We shall always suppose, without explicit statement, that $z$ is confined to the unit circle $|z| < 1$.

**Proof of** (2). To prove (2) we assume $a_p \neq 0$ and let

$$P(z) = \sum_{n=1}^{p} a_n z^n = a_p \prod_{n=1}^{p} (z - b_n), \tag{4}$$

from which we have

$$|P(z)| \leqq \sum_{n=1}^{p} |a_n| = C. \tag{5}$$

We observe that $|a_n| \leqq M$, so that $|P(z)| < pM$, and therefore by the Schwarz lemma,

$$|f(z) - P(z)| < (p+1)M|z|^{p+1}. \tag{6}$$

Now let

$$\begin{aligned} P_1(r) &= |a_p| \prod_{n=1}^{p} (r - |b_n|) = c_1 r + \cdots + c_p r^p, \\ C' &= \sum_{n=1}^{p} |c_n| = |a_p| \prod_{n=1}^{p} (1 + |b_n|). \end{aligned} \tag{7}$$

Then by (4) and (7), $|P(re^{i\theta})| \geqq |P_1(r)|$. Also, since $c_n$ is the same polynomial in the $|b_j|$ as $a_n$ is in the $b_j$, and since only plus signs (or only minus signs) occur, we have $|c_n| \geqq |a_n|$, and $C' \geqq C$. Together with (7) this gives

$$|P(re^{i\theta})| \geqq C \left| \prod_{n=1}^{p} \frac{r - |b_n|}{1 + |b_n|} \right|. \tag{8}$$

Note that if $|b_i| > 1$ we only strengthen the inequality (8) if we replace $|b_i|$ by 1 in the quotients occurring there. But then (8) implies that

$$|P(re^{i\theta})| \geqq C2^{-p}\prod_{n=1}^{p}|r - b_n'|, \tag{9}$$

where $b_n' = |b_n|$ if $|b_n| \leqq 1$, and $b_n' = 1$ otherwise. Now choose any positive real number $x_0 < 1$. In the interval $(0, x_0)$ there is at least one point $r_1$ such that for $i \leqq p$,

$$|r_1 - b_i'| \geqq x_0/2^p,$$

and therefore by (9),

$$|P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p}, \tag{10}$$

and by (6) and (10)

$$|P(r_1e^{i\theta})| - |f(r_1e^{i\theta}) - P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p} - x_0^{p+1}(p+1)M.$$

This expression is maximum for $x_0 = Cp/(4p)^p(p+1)^2M$, its maximum value being $C^{p+1}/M^pK^{p+1}$, where $K = (p+1)((p+1)/p)^{p/(p+1)}(4p)^p$.

The polynomial $P(z)$ has $n$ zeros in $|z| \leqq r_1$ with $n \leqq p$. Therefore by Rouché's theorem, $f(z)$ assumes in $|z| \leqq r_1$ every value in $|w| \leqq C^{p+1}/K^{p+1}M^p$ precisely $n$ times. Therefore $D_p(0) \geqq C^{p+1}/K^{p+1}M^p$; that is,

$$\sum_{n=1}^{p}|a_n| \leqq KM^{p/(p+1)}[D_p(0)]^{1/(p+1)}$$

as was to be proved.

In the above proof we have assumed that $a_p \neq 0$. If $a_p = 0$ the proof is valid for some $n \leqq p$, and $p$ can be reintroduced in the later stages, say in equation (8), where the product can be taken over $p$ factors if some of the $|b_n|$ are allowed to vanish.

**Proof of (3).** We prove (3) by considering the class of functions

$$f(z) \equiv Mz^p\frac{a - Mz}{M - az} \tag{11}$$

$$\equiv az^p\frac{1 - (Mz/a)}{1 - (az/M)} \tag{12}$$

where $a < M$. These functions have been studied by the author in a

previous note.[2] From (11) it is clear that $|f(z)| \leqq M$ in $|z| \leqq 1$, and that the bound is attained for $z=1$. It is also evident from (11) that the minimum of $|f(z)|$ on $|z| = r < a/M$ occurs at $z=r$. Thus $|f(z)| \geqq f(r)$ for $|z| = r < a/M$. By Rouché's theorem $f(z)$ covers the circle $|w| < f(r)$ precisely $p$ times in $|z| < r$, since $f(z)=0$ has $p$ roots in this circle. By Rolle's theorem $f'(z)$ vanishes for some $r_0$ between $r=0$ and $r=a/M$. Then every value of $|w| \leqq f(r_0)$ except $w=f(r_0)$ is assumed precisely $p$ times in $|z| \leqq r_0$, and $w=f(r_0)$ is assumed $p+1$ times. Therefore $D_p(0)=f(r_0)$. But by (12)

$$D_p(0) = f(r_0) < a(r_0)^p < a(a/M)^p = a^{p+1}/M^p. \tag{13}$$

Also by (12), the first term in the expansion of $f(z)$ is $az^p$ so that the sum of the moduli of the first $p$ coefficients is $a$. Thus by (13)

$$\sum_{n=1}^{p} |a_n| = a > M^{p/(p+1)}[D_p(0)]^{1/(p+1)}$$

which is the desired inequality (3).

**The coefficient $\lambda_p$.** We have left to prove that $\lambda_p$ can be taken as 1; in other words, that

$$D_p(0) \leqq \sum_{n=1}^{p} |a_n| = C.$$

As before, let $P(z)=a_1z+\cdots+a_pz^p$, and suppose that $D_p(0)>C$. Then $|f(z)| = C$ defines an analytic Jordan curve $\Gamma$ in $|z|<1$ and $\Gamma$ contains the origin in its interior. On $\Gamma$ we have $|P(z)| < C = |f(z)|$. Therefore by Rouché's theorem $f(z)-P(z)$ and $f(z)$ have the same number of zeros interior to $\Gamma$. But $f(z)$ has at most $p$ zeros inside $\Gamma$ since the image of the interior of $\Gamma$ under $f(z)$ is an $n$-sheeted circle with $n \leqq p$. And $f(z)-P(z) \equiv a_{p+1}z^{p+1}+\cdots$ has at least $p+1$ zeros interior to $\Gamma$. By this contradiction we see that $D_p(0) \leqq C$ as we wished to prove.

This result is trivially the best possible, for if $f(z) \equiv az^p$, then $D_p(0)=a=C=M$.

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[2] L. H. Loomis, *The radius and modulus of n-valence for analytic functions whose first n−1 derivatives vanish at a point*, this Bulletin, vol. 46 (1940), pp. 406–501.

Two problems were considered in this note. One was the problem of the present paper for the class of functions whose first non-vanishing coefficient is $a_p$. The second was the problem of determining for this restricted class of functions the largest circle $|z| < r$ (the radius depending on $p$, $M$ and $|a_p|$) in which all the functions are at most $p$-valent. The present note does not treat the corresponding problem for the more general class of functions considered here.

# THE CAUCHY THEOREM FOR FUNCTIONS ON CLOSED SETS

PHILIP T. MAKER

The object of this paper is to extend the theorem of Cauchy to functions of a complex variable defined on any bounded closed set, $E$, by determining conditions on $f(z)$ in order that for certain coverings of $E$, $C_n$, and an extension of $f(z)$, $f^*(z)$, $\lim_{n\to\infty} \int_{c_n} f^*(z)dz = 0$. It was suggested partly by the notion of a general monogenic function due to Trjitzinsky[1] and partly by the measure theory methods of Menchoff[2] and others, which succeed so well in lightening the restrictions on the real and imaginary parts of a complex function in order that $f(z)$ be regular.

Throughout this paper we shall consider only rectangles with sides parallel to the real and imaginary axes. A $C$-covering of a plane set $F$, denoted by $C$, will be a set of closed rectangles, possibly abutting, but nonoverlapping, which contain $F$. $c$ will denote the boundary of $C$. The covering $C_n$ is to be composed of rectangles $R_{mn}$ so that $C_n = \sum_m R_{mn}$ $(m, n = 1, 2, \cdots)$.

1. **The extension, $f^*(z)$.** If $u(P)$ is a positive continuous function defined on the closed and bounded set $F$ in the plane, we shall let[3] $u^*(P) = \max_{Q\in F} u(Q)\{2 - d(P, Q)/d(P, F)\}$ for $P$ not in $F$, and $u^*(P) = u(P)$ for $P$ in $F$, where $d(P, Q)$ denotes the distance from $P$ to $Q$ and $d(P, F)$ the distance from the set $F$ to $P$. In general, if $u(P)$ is continuous, since $u(P) = (u(P) + |u(P)|)/2 - (|u(P)| - u(P))/2$, that is, since $u(P)$ is the difference of two continuous positive functions, $u^*(P)$ will denote the extension of $u(P)$ obtained by extending as before these parts. If $f(z)$ $(= u(x, y) + iv(x, y))$ is defined on a bounded closed set and continuous, $f^*(z)$ will denote $u^*(x, y) + iv^*(x, y)$.

LEMMA 1. *If $u(P)$ is defined on a bounded closed set $F$ and $|u(Q) - u(P)| < M(P)d(P, Q)$ where $M(P)$ is a finite function of $P$ defined on $F$, then $|u^*(P) - u^*(Q)| < 20\, M(P)\, d(P, Q)$, for $P$ in $F$ and $Q$ arbitrary.*

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Presented to the Society, December 29, 1941; received by the editors February 25, 1942.

[1] W. J. Trjitzinsky, *Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux*, Annales Scientifique de L'École Normale Supérieure, Paris, 1938, p. 120.

[2] D. Menchoff, *Les Conditions de Monogénéité*, Actualités Scientifiques et Industrielles, no. 329, Paris, 1936.

[3] S. Bochner, *Fourier Lectures*, 1936–1937, Princeton, p. 62.

PROOF. Consider first the case for which $u(P) \geqq 0$. If $Q$ is any point not in $D$, if $Q_0$ is a point in $F$ for which $d(Q, Q_0) = d(Q, F)$, and if, of the points $S$ satisfying the inequality $d(S, Q) \leqq 2d(Q, Q_0)$, $R$ is the point where the maximum of $u$ is attained, then $u^*(Q_0) \leqq u^*(Q) \leqq u^*(R)$. Hence

$$\begin{aligned} |u^*(Q) - u^*(P)| &\leqq |u^*(Q) - u^*(R)| + |u^*(R) - u^*(P)| \\ &\leqq |u^*(Q_0) - u^*(P)| + 2|u^*(R) - u^*(P)| \\ &< M(P)d(Q_0, P) + 2M(P)d(R, P). \end{aligned}$$

It is easily verified that $d(Q_0, P) \leqq 2d(Q, P)$ and $d(R, P) \leqq 4d(Q, P)$, so that $|u^*(Q) - u^*(P)| < 10M(P)\, d(P, Q)$ and the lemma is proved for case of $u(P)$ positive. In the general case, $u(P) = (|u| + u)/2 - (|u| - u)/2 = g(P) - h(P)$, where $g$ and $h$ are positive functions, and satisfy the conditions of the lemma, so that for $P$ in $F$ and $Q$ arbitrary $|g^*(Q) - g^*(P)|$ and $|h^*(Q) - h^*(P)|$ are each less than $10M(P)\, d(P, Q)$. Hence for $u^* = g^* - h^*$, it readily follows that $|u^*(Q) - u^*(P)| < 20M(P)\, d(P, Q)$, and the proof of the lemma is complete.

**2. Bounded derivatives.** We shall use the following fundamental lemma:[4]

LEMMA 2. *Let $w(x, y)$ be a real, continuous function defined in the square $S$, the sides of which are parallel to the coordinate axes, and let $F$ be a closed set in $S$ and such that*

$$\begin{aligned} |w(x + h, y) - w(x, y)| &\leqq M|h|, \\ |w(x, y + k) - w(x, y)| &\leqq M|k| \end{aligned}$$

*for all points $(x, y)$ in $F$ and for all points $(x+h, y)$, $(x, y+k)$ of the square $S$, where $M$ is a constant. Finally let $R$ be the least rectangle with sides parallel to the axes containing $F$.*[5]

*Under these conditions*[6] *the following inequalities hold:*

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)]dx - \iint_F \frac{\partial w}{\partial y}\, dxdy \right| \leqq 5Mm(S - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)]dy - \iint_F \frac{\partial w}{\partial x}\, dxdy \right| \leqq 5Mm(S - F)$$

*where $(x_1, y_1)$, $(x_2, y_1)$, $(x_2, y_2)$ and $(x_1, y_2)$, $(x_1 \leqq x_2,\ y_1 \leqq y_2)$ are the*

[4] For the proof of this lemma, cf. loc. cit., p. 10.

[5] The "least rectangle" may be only a segment or a point.

[6] The conditions on $w$ imply the existence of the partial derivatives a.e. in $F$.

*corners of the rectangle $R$ and $m(S-F)$ is the measure of the set $S-F$, which is composed of points of $S$ not in $F$.*

THEOREM 1. *Let $f(z)$ $(=u(x, y)+iv(x, y))$ be defined on the bounded closed set $E$, and let $R$ be a rectangle with sides parallel to the axes containing at least one point of $E$ on each side. If (letting $F=E\cdot R$)*

(1) *for all $z$ and $z+h$ in $F$, and a constant $B$,*

$$\left|\frac{f(z+h)-f(z)}{h}\right| < B,$$

(2) *the Cauchy-Riemann equations hold a.e. (almost everywhere) in $F$, where the partial derivatives of $u$ and $v$ exist,*[7] *then $\left|\int_r f^*(z)dz\right| < 400Bm(S-F)$ where $r$ is the boundary of $R$, and $S$ is a square of least area containing $R$.*

PROOF. If $h=k+il$, condition (1) implies

$$\left|\frac{u(x+k, y+l)-u(x, y)}{h}\right| < B,$$

and a similar condition on $v(x, y)$, for every point $z$, and $z+h$, in $F$. According to Lemma 1,

$$\left|\frac{u^*(x+k, y+l)-u^*(x, y)}{h}\right| < 20B,$$

for each point $z$ in $F$, and $z+h$ in $R$. Hence by Lemma 2,

$$\left|\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx - \int\int_P \frac{\partial u^*}{\partial y}\,dxdy\right| < 100Bm(S-F),$$

$(x_1, y_1)$ and $(x_2, y_2)$ being corners of $R$. Similar inequalities for $u^*(x, y)$ with respect to $y$, and $v^*(x, y)$ with respect to $x$ and $y$ also hold. But

$$\int f^*(z)dz = \int_r u^*dx - v^*dy + i\int_r v^*dx + u^*dy$$

and

$$\int_r u^*dx = -\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx.$$

[7] This is in no way a further restriction on $E$, for almost all points of any measurable plane set are points of linear density for it both in the direction of the $x$-axis and in that of the $y$-axis.

The condition that the limits, $\lim_{h\to 0}(f(z+h)-f(z))/h$, as $z+h$ approaches $z$ through points of $E$ along either of two curves having non-collinear tangents at $z$, should be equal, is equivalent to the condition (2) in the presence of (1). For (1) and (2) imply that $f^*(z)$ is monogenic a.e. in $E$ (Menchoff, loc. cit., Theorem 2, p. 27 and Theorem 5, p. 23).

Let $\epsilon' = \int_{z_1}^{z_2} [u^*(x, y_2) - u^*(x, y_1)]\, dx - \iint_F (\partial u^*/\partial y) dy dx$. Then $\int_\gamma u^* dx = -\iint_F (\partial u^*/\partial y) dy dx - \epsilon'$. Taking into account similar reasoning for the other parts of $\int_\gamma f^*(z) dz$ we have

$$\int_\gamma f^*(z) dz = -\iint_F \left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}\right) dx dy + i \iint_F \left(\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y}\right) dx dy + \epsilon$$

where $|\epsilon| < 400Bm(S-F)$. Since the Cauchy-Riemann equations hold a.e. in $F$, $\int_\gamma f^*(z) dz = \epsilon$, and the proof of the theorem is complete.

COROLLARY. *Let $f(z)$ be defined on the bounded closed set $E$ with a bounded derivative there. Let $S = \sum_m S_m$ be a C-covering of $E$ by squares with $R_m$ the least rectangle within $S_m$ containing $S_m \cdot E$. Then there is a constant $B$ for which $\sum_m \left| \int_{\gamma_m} f^*(z) dz \right| < 400Bm(S-E)$, and if $S \to E$, $C = \sum_m R_m \to E$, and $\lim_{C \to E} \int_C f^*(z) dz = 0$.*

**3. Derivatives finite, except for a denumerable set.** We prove this theorem:

THEOREM 2. *If $f(z)$ is defined and continuous on the bounded closed set $E$, and if, except for a denumerable number of points, $\limsup_{h \to 0} |(f(z+h) - f(z))/h| < \infty$, and the Cauchy-Riemann equations hold a.e. where the partial derivatives of $u$ and $v$ exist, then there is a sequence of C-coverings, $\{C_n\}$, for which $\lim_{n \to \infty} \sum_m \left| \int_{\gamma_m} f^*(z) dz \right| = 0$.*

PROOF. Define $I(C) = \sum_m \left| \int_{\gamma_m} f^*(z) dz \right|$. If for every point $z$ of $E$ there is a neighborhood $N(z)$ such that for every closed subset of $E$ in $N$, there is a sequence of coverings $\{C_n\}$ for which $\lim_{n \to \infty} I(C_n) = 0$, then by the Heine-Borel theorem there exists a sequence of coverings of $E$ with the property mentioned in the theorem. The proof will be complete therefore, if we show that there is such a neighborhood for each point of $E$. Let $P$ be those points of $E$ such that in every neighborhood of $z$ there is a closed subset of $E$ for which there is no sequence of $C$-coverings, $\{C_n\}$, for which $\lim_{n \to \infty} I(C_n) = 0$. We shall assume that $P$ is not empty and show that this leads to an absurdity.

Let $P_m$ $(m = 1, 2, \cdots)$ be the points of $P$ for which each of the absolute values,

$$|u^*(x+k, y) - u^*(x, y)|, \quad |v^*(x+k, y) - v^*(x, y)|,$$
$$|u^*(x, y+k) - u^*(x, y)|, \quad |v^*(x, y+k) - v^*(x, y)|$$

is less than or equal to $m|k|$ for $|k| \leqq 1/m$, $k$ a real number. Since

$u^*$ and $v^*$ are continuous and $P$ is closed, $P_m$ is closed. Since at each point of $E$, except for a denumerable set $H$, the partial derivates are finite, $P=\sum_m P_m+P\cdot H$. By Baire's[8] theorem, there is an isolated point of $P$ in $H$, or for some integer $N$ there is a point $z_0$ in $P$, the center of a square $S$ which contains only points of $P$ which are in $P_N$. The former alternative is quickly dismissed as impossible; we proceed on the basis of the latter, and let $F$ be any closed subset of $E\cdot S$. Subdivide the sides of $S$ into $n$ equal parts, $n>2N$, and obtain the squares $S_j$ $(j=1, 2, \cdots, n^2)$. $\epsilon$ being given, choose $n$ so great that the squares $\overline{S}_j$ which contain points of $F\cdot P$ satisfy the inequality, $m(\sum_j \overline{S}_j - P\cdot S)<\epsilon/800N$. If $\overline{R}_j$ is the least rectangle containing $P\cdot\overline{S}_j$, and $\overline{C}$ is the covering $\sum_j \overline{R}_j$, by Theorem 1, $I(\overline{C})<400\,N\sum_j m(\overline{S}_j - P\cdot\overline{S}_j)<\epsilon/2$. Since $I(R)$ is a continuous function of $r$, $\overline{C}$ may be extended by the addition of more small rectangles, so that, if $C'$ is the new covering, $I(C')$ remains less than $\epsilon/2$, but so that the points of $F\cdot P$ are inner points of the covering. The part of $F$ not already covered (denote it by $G$) is such that its closure contains only points $z$ of $F$ for which there is some neighborhood $N(z)$ with the property that every closed subset of $F$ in $N$ can be $C$-covered, say by $C_n(z)$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty} I(C_n(z))=0$. Let $S(z)$ be a square with $z$ as center entirely within $N(z)$. Of these squares a finite number, $k$, cover $G$, and within each of these is a covering, $C(z)$, of $G$ for which $I(C(z))<\epsilon/2k$. Hence $G$ is $C$-covered by a covering $C$ for which $I(C)<\epsilon/2$. $F$ is therefore $C$-covered by $C+C'$ for which $I(C+C')<\epsilon$, so that $z_0$ cannot belong to $P$, contrary to assumption. This completes the proof of Theorem 2.

COROLLARY. *If $f(z)$, defined on the bounded closed set $E$ and continuous there, has a derivative at each point except at most a denumerable set, there is a sequence of $C$-coverings of $E$ with $E$ as their limit for which* $\lim_{n\to\infty}\int_{C_n} f^*(z)dz=0$.

DUKE UNIVERSITY

[8] S. Saks, *The Theory of the Integral*, New York, 1937, p. 54.

# ON VALUE REGIONS OF CONTINUED FRACTIONS

WALTER LEIGHTON AND W. J. THRON

The results of this paper are contained in the following theorem.

THEOREM 1. *If the elements $a_1, a_2, \cdots$ of the continued fraction*

$$1+\frac{a_1}{1+}\,\frac{a_2}{1+}\cdots \tag{1}$$

*lie in the parabola*

$$\rho \leqq \frac{2d(1-d)}{1-\cos\theta}, \qquad 1/2 \leqq d < 1, \tag{2}$$

*then the approximants of the continued fraction* (1) *lie in the hyperbolic region*

$$R > \frac{2d(1-d)}{1-2d+\cos\phi}, \qquad -\beta < \phi < \beta, \tag{3}$$

*where* $\beta = \arccos(2d-1)$. *If $z$ is any value on the boundary of the region* (3), *there exists one and only one continued fraction of the form* (1), *with elements in the parabola* (2), *converging to $z$, namely*:[1]

$$1+\frac{a}{1+}\,\frac{\bar{a}}{1+}\,\frac{a}{1+}\cdots \tag{4}$$

*where $a=(z-1)\bar{z}$ is a value on the boundary of the parabola* (2).

For the case $d=1/2$, Scott and Wall [3] determined the value region of the approximants and Paydon [1] established the uniqueness property of (4) for that case.

A convergence criterion due to Scott and Wall [2] insures the convergence of the continued fraction (1) if in addition to the conditions of Theorem 1 it is required that the series $\sum|b_n|$ diverges, where $b_1=1/a_1$, $b_{n+1}=1/a_{n+1}b_n$. The value of such a continued fraction must lie in the closure of the region (3). Finally it follows from Theorem 1 that all values in the region (3) are assumed by a continued fraction of the form (4). The following result is now seen to be a consequence of Theorem 1.

Presented to the Society, April 11, 1942; received by the editors December 7, 1941, and, in revised form, March 27, 1942.

[1] Here and elsewhere in the paper a bar over a number means the complex conjugate of the number.

THEOREM 2. *If the elements $a_n$ $(n \geqq 1)$ of the continued fraction* (1) *lie in the parabola* (2) *and if in addition $\sum |b_n|$ diverges, where $b_1 = 1/a_1$, $b_{n+1} = 1/a_{n+1}b_n$, the continued fraction* (1) *converges, and its value lies in the closure of the hyperbolic region* (3). *Further, every value in* (3) *is taken on by at least one convergent continued fraction* (1) *with elements in* (2).

We now prove the first part of Theorem 1, namely, that the approximants of (1) lie in the region (3) if the elements of the continued fraction lie in the parabola (2). We first note, that if $z$ lies in (3) then $w = re^{i\psi} = z - 1$ satisfies the relation

$$(5) \qquad r < \frac{-2d(1-d)}{1-2d+\cos\psi}, \qquad \beta < \psi < 2\pi - \beta;$$

$$r < \infty, \qquad -\beta < \psi < \beta.$$

The proof is by induction. $A_1/B_1 - 1 = a_1$ satisfies relation (5), if $a_1$ lies in the parabola (2), as

$$\frac{2d(1-d)}{1-\cos\phi} \leqq \frac{-2d(1-d)}{1-2d+\cos\phi}$$

for $\beta < \phi < 2\pi - \beta$. Hence $A_1/B_1$ lies in the region (4). To complete the proof it is sufficient to show that $w = z - 1 = a/z'$ lies in the region defined by relation (5) if $a$ and $z'$ are any arbitrary complex numbers in the regions (2) and (3), respectively. To this end we prove the following lemma.

LEMMA. *The number*

$$w = \frac{a}{z} = \frac{\rho e^{i\phi}}{Re^{i\theta}} = \frac{\rho}{R}\, e^{i(\phi-\theta)}$$

*satisfies relation* (5), *if $a$ and $z$ are arbitrary complex numbers in the regions defined by relations* (2) *and* (3), *respectively.*

We shall choose $a$ and $z$ in such a way, that they maximize $|w|$ for any arbitrary fixed $\arg w = \alpha$, $\beta < \alpha < 2\pi - \beta$. It is clear that the numbers $a$ and $z$ must then be chosen on the boundaries of the regions over which they are allowed to vary, and we then have

$$|w| = \frac{\rho}{R} = \frac{1-2d+\cos\theta}{1-\cos\phi}, \qquad \phi - \theta = \alpha.$$

It follows from elementary considerations that if $|w|$ is to be a maximum, $\theta$ must satisfy the relation

$$\sin\theta + \sin\alpha + (1 - 2d)\sin(\alpha + \theta) = 0.$$

This is equivalent to

$$2\sin\frac{\alpha+\theta}{2}\left[\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2}\right] = 0.$$

The solution $\theta = -\alpha$ is excluded since the ranges of $\alpha$ and $-\theta$ have no values in common. It remains to find the value of $|w|$ when

$$\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2} = 0. \tag{6}$$

We shall show that (6) is equivalent to the two relations $|w| = -2d(1-d)/(1-2d+\cos\alpha)$ and $z = 1+\bar{w}$. If these two relations are satisfied the sine law leads to

$$\frac{\sin(\pi-\alpha)}{\sin(-\theta)} = \frac{\sin\alpha}{-\sin\theta} = -\frac{1-2d+\cos\alpha}{1-2d+\cos\theta}.$$

This gives

$$(1-2d)(\sin\alpha - \sin\theta) + \sin(\alpha-\theta) = 0,$$

or

$$2\sin\frac{\alpha-\theta}{2}\left[\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2}\right] = 0.$$

Since $\theta$ cannot equal $\alpha$, the only solution of this equation is that of equation (6). Thus the lemma is proved.

This completes the proof of the first part of Theorem 1. It remains to show that any $z$ on the boundary of (3) is uniquely expressible as a continued fraction of the form (4).

From the proof of the lemma it follows that $a_1$ must lie on the boundary of (2) for $z$ to lie on the boundary of (3). It was further shown that

$$z = 1 + \frac{a_1}{1+}\,\frac{a_2}{1+}\cdots$$

lies on the boundary only if

$$\bar{z} = 1 + \frac{a_2}{1+}\,\frac{a_3}{1+}\cdots;$$

but then $\bar{z}$ lies on the boundary and hence we must have

$$z = 1 + \frac{a_{2n-1}}{\bar{z}}, \qquad \bar{z} = 1 + \frac{a_{2n}}{z}, \qquad n \geqq 1.$$

This gives

$$a_{2n-1} = (z - 1)\bar{z},$$

$$a_{2n} = (\bar{z} - 1)z = \bar{a}_{2n-1},$$

and it is easily seen that all $a_n$ lie on the boundary of the parabola. The theorem is now completely proved.

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3. ———, *Value regions for continued fractions*, this Bulletin, vol. 47 (1941), pp. 580–585.

THE RICE INSTITUTE

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# A TABLE OF COEFFICIENTS FOR NUMERICAL DIFFERENTIATION

ARNOLD N. LOWAN, HERBERT E. SALZER AND ABRAHAM HILLMAN[1]

The following table lists the coefficients $A_{m,s}$ for $m = 1, 2, \cdots, 20$ and $s = m, \cdots, 20$ in Markoff's formula for the $m$th derivative in terms of advancing differences, namely

$$\omega^m f^{(m)}(x) = \sum_{s=m}^{n-1} (-1)^{m+s} A_{m,s} \Delta^s f(x) + (-1)^{m+n} \omega^n A_{m,n} f^{(n)}(\xi).$$

In this formula $\omega$ is the tabular interval and

$$A_{m,s} = (-1)^{m+s} m B_{s-m}^{(s)} / s(s-m)!$$

and $B_{s-m}^{(s)}$ is the $(s-m)$th Bernoulli number of the $s$th order.

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Presented to the Society, April 4, 1942 under the title *Coefficients of differences in the expansion of derivatives in terms of advancing differences*; received by the editors March 7, 1942.

[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project, Work Projects Administration, New York City.

If a function has been tabulated to sufficiently great accuracy and for some suitable interval of the argument along the real axis, the accompanying table may be used to generate the values of the derivatives which in turn may be employed to generate the values of the function in the complex plane within a region where the function is analytic.

The coefficients were computed from the recurrence formula

$$sA_{m,s} = (s-1)A_{m,s-1} + mA_{m-1,s-1}$$

and checked by independent calculations using the identity

$$x(x+1)(x+2)\cdots(x+s-1) \equiv s!\sum_{j=1}^{s} A_{j,s}x^j/j!.$$

From the identity

$$(x+x^2/2+x^3/3+\cdots)^m \equiv A_{m,m}x^m + A_{m,m+1}x^{m+1} + \cdots$$

it was discovered that a prime $p$ is not effectively present in the denominator of an $A_{m,s}$ for which $s<m+p-1$. The cancellation of prime factors in accordance with this rule was a further check on the accuracy of the work.

The Markoff formula is used at the beginning and end of a table where advancing differences are the only types available. For a full discussion see L. M. Milne-Thomson, *The Calculus of Finite Differences*, chap. 7, pp. 157–159. According to Milne-Thomson the relative simplicity of the remainder term is another advantage over central difference formulae.

Comparison of the Markoff coefficients with central difference coefficients shows the latter to be much smaller and obviously more convenient for obtaining the derivatives of a polynomial sufficiently far away from the ends of a table. However for many important functions in applied mathematics such as Bessel, error, and gamma functions, use of the Markoff formula for a polynomial approximation of some fixed degree might yield a smaller total error due to the particular form of its remainder term.

The first few coefficients of the various formulae may be found in H. T. Davis, *Table of the Higher Mathematical Functions*, vol. 1, pp. 73–77; Whittaker and Robinson, *Calculus of Observations*, pp. 62–65, and in an article by W. S. Bickley *Numerical differentiation near the limits of a difference table*, Philosophical Magazine, (7), vol. 33 (1942), pp. 12–14. (This article lists coefficients of the first 12 derivatives up to those of the 12th difference.)

COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION

| $m$ \ $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{1}{10}$ | $\frac{1}{11}$ | $\frac{1}{12}$ | $\frac{1}{13}$ | $\frac{1}{14}$ |
| 2 | | 1 | 1 | $\frac{11}{12}$ | $\frac{5}{6}$ | $\frac{137}{180}$ | $\frac{7}{10}$ | $\frac{363}{560}$ | $\frac{761}{1260}$ | $\frac{7129}{12600}$ | $\frac{671}{1260}$ | $\frac{83711}{166320}$ | $\frac{6617}{13860}$ | $\frac{1145993}{2522520}$ |
| 3 | | | 1 | $\frac{3}{2}$ | $\frac{7}{4}$ | $\frac{15}{8}$ | $\frac{29}{15}$ | $\frac{469}{240}$ | $\frac{29531}{15120}$ | $\frac{1303}{672}$ | $\frac{16103}{8400}$ | $\frac{190553}{100800}$ | $\frac{128977}{69300}$ | $\frac{9061}{4950}$ |
| 4 | | | | 1 | 2 | $\frac{17}{6}$ | $\frac{7}{2}$ | $\frac{967}{240}$ | $\frac{89}{20}$ | $\frac{4523}{945}$ | $\frac{7645}{1512}$ | $\frac{341747}{64800}$ | $\frac{412009}{75600}$ | $\frac{9301169}{1663200}$ |
| 5 | | | | | 1 | $\frac{5}{2}$ | $\frac{25}{6}$ | $\frac{35}{6}$ | $\frac{1069}{144}$ | $\frac{285}{32}$ | $\frac{31063}{3024}$ | $\frac{139381}{12096}$ | $\frac{1148963}{90720}$ | $\frac{355277}{25920}$ |
| 6 | | | | | | 1 | 3 | $\frac{23}{4}$ | 9 | $\frac{3013}{240}$ | $\frac{781}{48}$ | $\frac{242537}{12096}$ | $\frac{48035}{2016}$ | $\frac{1666393}{60480}$ |
| 7 | | | | | | | 1 | $\frac{7}{2}$ | $\frac{91}{12}$ | $\frac{105}{8}$ | $\frac{4781}{240}$ | $\frac{13321}{480}$ | $\frac{314617}{8640}$ | $\frac{790153}{17280}$ |
| 8 | | | | | | | | 1 | 4 | $\frac{29}{3}$ | $\frac{55}{3}$ | $\frac{10831}{360}$ | $\frac{897}{20}$ | $\frac{944311}{15120}$ |
| 9 | | | | | | | | | 1 | $\frac{9}{2}$ | 12 | $\frac{99}{4}$ | $\frac{1747}{40}$ | $\frac{5551}{80}$ |
| 10 | | | | | | | | | | 1 | 5 | $\frac{175}{12}$ | $\frac{65}{2}$ | $\frac{491}{8}$ |
| 11 | | | | | | | | | | | 1 | $\frac{11}{2}$ | $\frac{209}{12}$ | $\frac{1001}{24}$ |
| 12 | | | | | | | | | | | | 1 | 6 | $\frac{41}{2}$ |
| 13 | | | | | | | | | | | | | 1 | $\frac{13}{2}$ |
| 14 | | | | | | | | | | | | | | 1 |

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$\omega^m D^m f(x) \sim \sum_{s=m}^{\infty} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

| 15 | 16 | 17 | 18 | 19 | 20 | $s$ / $m$ |
|---|---|---|---|---|---|---|
| $\frac{1}{15}$ | $\frac{1}{16}$ | $\frac{1}{17}$ | $\frac{1}{18}$ | $\frac{1}{19}$ | $\frac{1}{20}$ | 1 |
| $\frac{1171733}{2702700}$ | $\frac{1195757}{2882880}$ | $\frac{143327}{360360}$ | $\frac{42142223}{110270160}$ | $\frac{751279}{2042040}$ | $\frac{275295799}{775975200}$ | 2 |
| $\frac{30946717}{17199000}$ | $\frac{39646461}{22422400}$ | $\frac{58433327}{33633600}$ | $\frac{344499373}{201801600}$ | $\frac{784809203}{467812800}$ | $\frac{169704792667}{102918816000}$ | 3 |
| $\frac{406841}{71280}$ | $\frac{35118025721}{6054048000}$ | $\frac{4446371981}{756756000}$ | $\frac{80847323107}{13621608000}$ | $\frac{2263547729}{378378000}$ | $\frac{32262100943}{5360355000}$ | 4 |
| $\frac{21939781}{1496880}$ | $\frac{2065639}{133056}$ | $\frac{2195261857}{134534400}$ | $\frac{371446039969}{21794572800}$ | $\frac{27566944753}{1556755200}$ | $\frac{31938836201}{1743565824}$ | 5 |
| $\frac{22463}{720}$ | $\frac{277382447}{7983360}$ | $\frac{38101097}{997920}$ | $\frac{1356664151597}{32691859200}$ | $\frac{162356544377}{3632428800}$ | $\frac{694142313941}{14529715200}$ | 6 |
| $\frac{899683}{16200}$ | $\frac{2271089}{34560}$ | $\frac{86853967}{1140480}$ | $\frac{13195009}{152064}$ | $\frac{227663026369}{2335132800}$ | $\frac{2022480780283}{18681062400}$ | 7 |
| $\frac{35717}{432}$ | $\frac{54576553}{518400}$ | $\frac{8424673}{64800}$ | $\frac{334947281}{2138400}$ | $\frac{9764119}{52800}$ | $\frac{5013017410969}{23351328000}$ | 8 |
| $\frac{515261}{5040}$ | $\frac{23915}{168}$ | $\frac{76492463}{403200}$ | $\frac{21878439}{89600}$ | $\frac{4065163957}{13305600}$ | $\frac{3975325483}{10644480}$ | 9 |
| $\frac{2485}{24}$ | $\frac{324509}{2016}$ | $\frac{59279}{252}$ | $\frac{79243781}{241920}$ | $\frac{11795941}{26880}$ | $\frac{6063698587}{10644480}$ | 10 |
| $\frac{30217}{360}$ | $\frac{1199}{8}$ | $\frac{494351}{2016}$ | $\frac{1513391}{4032}$ | $\frac{18843187}{34560}$ | $\frac{367394203}{483840}$ | 11 |
| $\frac{105}{2}$ | $\frac{26921}{240}$ | $\frac{6341}{30}$ | $\frac{5490071}{15120}$ | $\frac{976163}{1680}$ | $\frac{354467473}{403200}$ | 12 |
| $\frac{143}{6}$ | 65 | $\frac{35269}{240}$ | $\frac{46631}{160}$ | $\frac{3965533}{7560}$ | $\frac{10596053}{12096}$ | 13 |
| 7 | $\frac{329}{12}$ | $\frac{238}{3}$ | $\frac{136241}{720}$ | $\frac{31521}{80}$ | $\frac{6406481}{8640}$ | 14 |

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COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION
$\omega_m D^m f(x) \sim \sum_{s=m}^{s=n} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

| m \ s | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|
| 15 | 1 | $\frac{15}{2}$ | $\frac{125}{4}$ | $\frac{765}{8}$ | $\frac{11519}{48}$ | $\frac{50255}{96}$ |
| 16 | | 1 | 8 | $\frac{106}{3}$ | 114 | $\frac{18017}{60}$ |
| 17 | | | 1 | $\frac{17}{2}$ | $\frac{119}{3}$ | $\frac{1615}{12}$ |
| 18 | | | | 1 | 9 | $\frac{117}{4}$ |
| 19 | | | | | 1 | $\frac{19}{2}$ |
| 20 | | | | | | 1 |

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# ON MAJORANTS OF SUBHARMONIC AND ANALYTIC FUNCTIONS

FRANTIŠEK WOLF

This paper represents a different approach to a whole group of problems connected with majorants of subharmonic functions. The same method has been used previously in order to prove a generalization of the Phragmén-Lindelöf theorem.[1] It seems that the best approach is to prove first Lemma 4, and then the most important results are easily deducible. Corollary 6 is a generalization of a result of N. Levinson.[2] His theorem has made me realize the importance of these results.

LEMMA 1. *If* (i) $0<f(x)\leqq 1$ *and* (ii) $\int_a^b \log f(x)\cdot dx$ *is finite, then*

$$\int_a^b \log\left|\int_\xi^x f(y)dy\right| dx \tag{1}$$

*is a continuous function of* $\xi$ *in* $(a, b)$.

We first suppose that $f(x)$ is non-decreasing and that $(0, 1)=(a, b)$. We get

$$\int_0^x f(y)dy > \int_{x/2}^x f(y)dy \geqq (x/2)f(x/2).$$

Hence

$$\int_0^1 \log\left(\int_0^x f(y)dy\right)dx > \int_0^1 \log (x/2)dx + \int_0^1 \log f(x/2)dx$$

$$> 2\int_0^1 \log x\cdot dx + 2\int_0^1 \log f(x)dx \tag{2}$$

$$= -2 + 2\int_0^1 \log f(x)\cdot dx.$$

If $f(x)$ is replaced by $f(a+(b-a)x)$, we obtain

Presented to the Society, November 22, 1941 under the title *On majorants of analytic functions*; received by the editors March 2, 1942.

[1] Cf. the end of this paper and Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[2] *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, p. 127, Theorem 43.

$$\int_a^b \log\left(\int_a^x f(y)dy\right)dx \geqq -2(b-a) + (b-a)\log(b-a) + 2\int_a^b \log f(x)\cdot dx. \tag{3}$$

Next, if $f(x)$ is general, we form the rearranged non-decreasing function $\bar{f}(x)$ for which

$$\operatorname{meas}_x^E [f(x) \leqq y] = \operatorname{meas}_x^E [\bar{f}(x) \leqq y]$$

for all $y$. We know that[3]

$$\int_0^1 \log f(x)\cdot dx = \int_0^1 \log \bar{f}(x)\cdot dx,$$

and that

$$\left|\int_{x_0}^x f(y)\cdot dy\right| \leqq \int_0^{|x-x_0|} \bar{f}(y)\cdot dy.$$

Hence, if $x_0 \subset (0, 1)$,

$$\begin{aligned}
&\int_0^1 \log\left|\int_{x_0}^x f(y)\cdot dy\right|\cdot dx \\
&\quad\geqq \int_0^1 \log\left(\int_0^{|x-x_0|} \bar{f}(y)\cdot dy\right)\cdot dx \\
&\quad= \int_0^{x_0} \log\left(\int_0^{x_0-x} \bar{f}(y)\cdot dy\right)\cdot dx + \int_{x_0}^1 \log\left(\int_0^{x-x_0} \bar{f}(y)\cdot dy\right)dx \\
&\quad= \int_0^{x_0} \log\left(\int_0^{x} \bar{f}(y)\cdot dy\right)\cdot dx + \int_0^{1-x_0} \log\left(\int_0^{x} \bar{f}(y)dy\right)dx \\
&\quad\geqq 2\int_0^1 \log\left(\int_0^x \bar{f}(y)\cdot dy\right)\cdot dx.
\end{aligned}$$

This, with (2) gives

$$\begin{aligned}
\int_0^1 \log\left(\int_{x_0}^x f(y)\cdot dy\right)\cdot dx &\geqq -4 + 4\int_0^1 \log \bar{f}(x)\cdot dx \\
&= -4 + 4\int_0^1 \log f(x)\cdot dx.
\end{aligned}$$

[3] Hardy, Littlewood, Pólya, *Inequalities*, Cambridge, 1934, p. 276.

In a similar fashion we get, for $x_0 \subset (a, b)$,

$$\int_a^b \log \left| \int_{x_0}^x f(y)dy \right| dx \geqq -4(b-a) + 2(b-a)\log(b-a) + 4\int_a^b \log f(x)\cdot dx. \tag{4}$$

If $\epsilon > 0$ is given, we can find a $\delta > 0$ such that

$$8\delta - 4\delta \log \delta - 4\int_{x_0-\delta}^{x_0+\delta} \log f(x)\cdot dx < \epsilon/3.$$

Let us denote by $J_1$ the common part of $(a, b)$ and $(x_0-\delta, x_0+\delta)$, and by $J_2$ the rest of $(a, b)$. Then, by (3), for $\xi \subset J_1$

$$0 > \int_{J_1} \log \left| \int_\xi^x f(y)\cdot dy \right| \cdot dx \geqq -8\delta + 4\delta \log \delta + 4\int_{x_0-\delta}^{x_0+\delta} \log f(x)\cdot dx > -\epsilon/3.$$

Further

$$g_2(\xi) = \int_{J_2} \log \left| \int_\xi^x f(y)\cdot dy \right| \cdot dx$$

is a continuous function for $\xi = x_0$ and, hence, there is a $\delta_1 < \delta$ such that

$$|g_2(\xi) - g_2(x_0)| \leqq \epsilon/3 \qquad \text{for } |\xi - x_0| < \delta_1.$$

Hence, if we call the integral (1) $g(\xi)$, then

$$|g(\xi) - g(x_0)| < \epsilon \qquad \text{for } |\xi - x_0| < \delta_1.$$

This shows that (1) is a continuous function of $\xi$ at an arbitrary point $x_0 \subset (a, b)$.

LEMMA 2. *Given a non-negative $\psi(x) \subset L$, there is a domain $D$, bounded by two continuous curves*

$$C_1 \equiv y = g_1(x), \qquad C_2 \equiv y = g_2(x)$$

*and two straight lines $x = x_0 - a$, $x = x_0 + a$, such that, if $x + iy = f(re^{i\theta})$ represents $D$, conformally on the unit circle $r < 1$, then*

$$\left| \frac{dx}{d\theta} \right|_{r=1} \geqq \text{const. } \exp[\psi(x)] \tag{5}$$

*on $C_1$ and $C_2$.*

*Further the rectangle*

$$R_0\colon [|y-y_0|<a\cdot\log 3/2,\ |x-x_0|<a] \tag{6}$$

*is interior to D and*

$$|y-y_0|\leqq c(a),\quad |x-x_0|\leqq a,\qquad \textit{with } \lim_{a\to 0} c(a)=0,$$

*contains D.*

There is no loss of generality in supposing $x_0=y_0=0$. We define $D$ by constructing its conformal representation on the unit circle. First we define the boundary function of the harmonic function $x(r,\theta)$ so that it satisfies the above condition for the derivative. We define it as the inverse function of

$$\theta(x)=b\int_{-a}^{x}e^{-\psi(t)}dt \tag{7}$$

and, for later convenience, we take $b$ such that

$$b\int_{-a}^{a}e^{-\psi(t)}dt=\theta_0<2\arccos 3/4<\pi/3. \tag{8}$$

Then the inverse function $x(1,\theta)$, defined in $(0,\theta_0)$ satisfies condition (5) and is continuous and decreasing. Further, we define $x(1,\theta)=a$, for $\theta\subset(\theta_0,\pi)$; $x(1,\pi+\theta)=x(1,\theta_0-\theta)$ for $\theta\subset(0,\theta_0)$, and finally $x(1,\theta)=-a$ for $\theta\subset(\pi+\theta_0,2\pi)$. In the interval $(\pi,\pi+\theta_0)$ condition (5) is again satisfied. Now, the boundary function is completely defined and

$$x(r,\theta)=\frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{x(1,\phi)d\phi}{1-2r\cos(\theta-\phi)+r^2}.$$

The conjugate harmonic function is

$$y(r,\theta)=-\frac{1}{2\pi}\int_0^{2\pi}\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}x(1,\phi)d\phi.$$

By partial integration

$$y(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\log\left|1-re^{i(\theta-\phi)}\right|dx(1,\phi).$$

Since $x$ is constant in some intervals, we get, using the above definition of $x(1,\phi)$,

$$(9)\qquad y(r,\theta)=\frac{1}{2\pi}\int_0^{\theta_0}[\log\left|1-re^{i(\theta-\phi)}\right| - \log\left|1+re^{i(\theta-\theta_0+\phi)}\right|]dx(1,\phi).$$

Now, the required conformal representation is given by $x(r,\theta)+iy(r,\theta)$. The boundary curve of $D$ in the parametric form is $x(1,\theta)+iy(1,\theta)$, and we shall investigate it.

If $\theta\subset(0,\theta_0)$, then $x(1,\theta)$ runs from $-a$ to $+a$ and, since $|1-e^{i(\theta-\phi)}|<1$ for $\theta,\phi\in(0,\theta_0)\subset(0,\pi/3)$ (cf. (8)),

$$\begin{aligned} y(1,\theta) &\leqq -\frac{1}{2\pi}\int_0^{\theta_0}\log\left|1+e^{i(\theta-\theta_0+\phi)}\right|dx \\ &\leqq -\frac{1}{2\pi}\log\left|1+e^{i\theta_0}\right|\int_{-a}^{a}dx \\ &= -(a/\pi)\log 2\cos\theta_0/2 < -(a/\pi)\log 3/2. \end{aligned}$$

The first term $I_1$ on the right in (9) is more difficult to dispose of. We have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_0^{\theta_0}\log\left|2\sin\frac{\theta-\phi}{2}\right|\cdot dx(1,\phi) \\ &= \frac{1}{2\pi}\int_0^{\theta_0}\log\frac{2\sin(\theta-\phi)/2}{\theta-\phi}\cdot dx(1,\phi) \\ &\quad+\frac{1}{2\pi}\int_0^{\theta_0}\log\left|\theta-\phi\right|\cdot dx(1,\phi) = I_1'+I_1'', \end{aligned}$$

Here, $I_1'$ is evidently a continuous function of $\theta$ and

$$0>I_1>-\frac{a}{\pi}\log\frac{2\sin\theta_0/2}{\theta_0}.$$

By means of (7), we change variables in $I_1''$ and obtain

$$I_1''=\frac{1}{2\pi}\int_{-a}^{a}\log\left|b\int_{\xi}^{x(\theta)}e^{-\psi(\alpha)}\cdot d\alpha\right|\cdot d\xi.$$

By Lemma 2, this is a continuous function of $x$ and therefore also of $\theta$ and from (3) we get

$$0>I_1''>-\frac{2a}{\pi}+\frac{a\log 2a}{\pi}+\frac{1}{\pi}\int_{-a}^{a}\log\left|be^{-\psi(x)}\right|dx.$$

If we combine the last inequalities and introduce the abbreviation $c(a)$, we get

$$-\frac{a}{\pi}\log 3/2 > y(1,\theta) > -c(a), \qquad \theta \subset (0,\theta_0),$$

with

$$\lim_{a\to 0} c(a) = 0.$$

From (9) it is easy to deduce that

$$y(r,\theta+\pi) = -y(r,\theta_0-\theta),$$

and we obtain

$$(a/\pi)\log 3/2 < y(1,\theta) \leqq c(a), \text{ for } \theta \subset (\pi,\pi+\theta_0).$$

Using the same notation we shall prove the following lemma.

LEMMA 3. *If $\psi(x_0-a)$, $\psi(x_0+a)$ are finite, then every subharmonic function $\alpha(x,y)$, defined in $D$, which satisfies*

$$\alpha(x,y) \leqq e^{\psi(x)}, \qquad |x| < a, \tag{10}$$

*has an upper bound in $R[|y| < a\cdot\log 3/2,\ |x| < a]$ which depends only on $\psi(x)$.*

If we represent $D$ conformally on the unit circle $C$, we get from $\alpha(x,y)$ a subharmonic function $\alpha(r,\theta)$ defined in $C$. There, it must be less than any harmonic function with boundary values not less than those of $\alpha(r,\theta)$. By (10), these are not greater than $\exp\{\psi(x(1,\theta))\}$, where $x(1,\theta)$ has been defined in the preceding lemma. Such a harmonic function is the Poisson integral of $\exp\{\psi(x(1,\theta))\}$. We have to show that this Poisson integral is not identically equal to $\infty$. A sufficient condition is

$$\int_0^{2\pi} \exp\{\psi(x(1,\theta))\}\cdot d\theta < \infty.$$

In view of the definition of $x(1,\theta)$, this integral is obviously less than

$$\left|\int_{-a}^{a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\theta_0}^{\pi} e^{\psi(a)}d\theta + \left|\int_{a}^{-a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\pi+\theta_0}^{2\pi} e^{\psi(-a)}d\theta,$$

and using (5), also less than

$$2b\int_{-a}^{a} e^{\psi(x)}e^{-\psi(x)}dx + \pi(e^{\psi(a)}+e^{\psi(-a)}) \leqq 4ab + \pi(e^{\psi(a)}+e^{\psi(-a)}) < \infty.$$

Hence $\alpha(x, y)$ is, inside of $D$, less than a finite harmonic function depending only on $\psi(x)$. In the domain $D$, completely interior to $R$, it is therefore bounded from above by a constant depending only on $\psi(x)$.

THEOREM. *If* (i) $\alpha(x, y)$ *is subharmonic in* $R[|x|<c, |y|<d]$ *and*

(ii) $\alpha(x, y)<e^{\psi(x)}$, $\psi(x)\subset L$, $x\subset(-c, c)$, $\psi(-c)<\infty$, $\psi(c)<\infty$,

*then for any* $\delta$, *such that* $0<\delta<d$, *there is an upper bound* $C$ *for* $\alpha(x, y)$ *in* $D_0[|x|<c, |y|<d-\delta]$ *dependent only on* $\delta$ *and* $\psi(x)$, *but independent of the particular* $\alpha(x, y)$.

In view of the Lemma 3, it is sufficient to show that there is an $a$ and a finite number of domains $D_k$ satisfying the conditions of this lemma, and, such that $D_k$ contains (cf. (6)) $R_k[|x-x_k|<a, |y-y_k|\leqq c(a)]$, is contained in $R$, and that $\sum R_k$ covers completely $D_0$. Since $\alpha(x, y)$ has a finite upper bound in every $R_k$, it must be so also in $D_0$. And the upper bound will depend only on $\psi(x)$ and $D_0$.

We determine $a>0$ such that $c(a)-a\log 3/2<\delta/2$. If we define $D^*[|x|<c, |y|<d-\delta/2]$, then $D_0\subset D^*\subset R$, and, if $R_k\subset D^*$, then the corresponding $D_k\subset R$ (cf. (3.2)). Since $R_k$ can be any rectangle, of the above size, in $D^*$ and such that $\psi(x_k\pm a)$ is finite, it is evident that we can find a finite number of them covering completely $D_0$. The theorem is proved.

COROLLARY: *Let* $f(z)$ *be a function, analytic in* $R[|x|<c, |y|<d]$, *such that*

$$|f(z)|\leqq M(x).$$

*If* $\int_{-\infty}^{\infty}\log^+\log^+ M(x)dx<\infty$, *then for every domain* $D_0$ *completely interior to* $R$, *there exists a* $\phi$ *depending only on* $D_0$ *and* $M(x)$, *such that*

$$|f(z)|\leqq\phi \qquad \text{for } z\subset D_0.$$

There is no loss of generality to suppose $M(x)>e$. and then the sign $+$ can be omitted over the log signs. If $f(z)$ is analytic, then $\log|f(z)|$ is subharmonic and the result follows from the preceding theorem.

Nils Sjöberg has proved the following theorem:[4]

*Let* $M(\theta)$ *be given and* $0<\epsilon<1$. *In order that the class of subharmonic functions, defined in* $|z|<1$, *which satisfy in* $1-\epsilon\leqq|z|<1$

$$|\mu(re^{i\theta})|\leqq M(\theta),$$

[4] Comptes Rendus du Congrès des Mathématiques à Helsinfors, 1938.

*should be bounded from above in every circle* $|z| \leqq r_0 < 1$, *it is sufficient that*

$$\int_{-\pi}^{\pi} \log^+ M(\theta) \cdot d\theta < \infty.$$

This theorem can easily be deduced from our result. By $\zeta = \log z$, we straighten out the concentric circles, and $1-\epsilon \leqq |z| < 1$ corresponds to $\log(1-\epsilon) \leqq R(\zeta) < 0$. In this strip $\mu(\zeta) \leqq M(\theta)$, $\theta = \Im(\zeta)$. Hence, if $M(\theta_0) < \infty$, then, by our result, the class of subharmonic functions $\mu(\zeta)$ will be bounded in $\theta_0 \leqq \arg z = \theta \leqq \theta_0 + 2\pi$ and $\log(1-\epsilon/2) \leqq R(\zeta) \leqq \log(1-\epsilon/4)$. Hence the same is true of $\mu(z)$ in $1-\epsilon/2 \leqq |z| \leqq 1-\epsilon/4$. But the class of subharmonic functions must have the same upper bound in $|z| \leqq 1-\epsilon/4$. The theorem is proved.

Similarly we can prove a generalization of the Phragmén-Lindelöf theorem:[5]

*If*: (i) $f(z)$ *is analytic in* $\Im(z) > 0$, (ii) *its boundary values on* $\Im(z) = 0$ *are in absolute value less than* 1, (iii) *there are two sequences* $r_k \to \infty$ *and* $\epsilon_k \to 0$ *such that*

$$|f(z)| \leqq \exp\{\epsilon_k r_k e^{\psi(\theta)}\}, \qquad \psi(\theta) \subset L,$$

*for* $r_k(1-\delta) < |z| < r_k$, *then*

$$|f(z)| \leqq 1 \qquad \text{for } \Im(z) > 0.$$

From condition (iii) it follows that

$$\log |f(r_k z)| / \epsilon_k r_k \leqq e^{\psi(\theta)},$$

for $1-\delta < |z| < 1$. By condition (ii), we can suppose $\psi(0) = \psi(\pi) = 0$. In the same way as in the preceding we deduce the existence of a $\phi$, such that

$$\log |f(r_k z)| / \epsilon_k r_k < \phi \qquad \text{for } 1 - \delta/2 < |z| < 1 - \delta/4,$$

or

$$|f(z)| \leqq \exp\{\epsilon_k \phi r_k\}, \qquad (1 - \delta/2) r_k < |z| < (1 - \delta/4) r_k.$$

Now we use the Phragmén-Lindelöf theorem in its classical form[6] to deduce the desired result.

UNIVERSITY OF CALIFORNIA

[5] Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[6] R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936, p. 43.

# HOMOGENEOUS AND NONHOMOGENEOUS DIOPHANTINE EQUATIONS

ANTHONY A. AUCOIN

In an earlier paper[1] we considered homogeneous polynomials $f$ and $g$ whose degrees are relatively prime, and solved the Diophantine equation $f(x)=g(y)$. These results are generalized in the present paper. The solutions are given in terms of arbitrary parameters and if the parameters are integral the solutions are also.

We begin our discussion with the hypothesis that the functions $f(x)=f(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$, $g(x)=g(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$ are polynomials with integral coefficients, homogeneous in each set of variables $x_{k1}x_{k2}\cdots x_{km}$; $f$ being of degree $\alpha_k \geqq 0$, $g$ being of degree $\beta_k \geqq 0$ in the sets $x_{k1}x_{k2}\cdots x_{km}$ and $d_k=\alpha_k-\beta_k$. We suppose further that integers $\lambda_i$, $\mu_i \geqq 0$ exist such that[2]

$$\sum_{i=1}^{n} d_i\lambda_i = -\sum_{i=1}^{n} d_i\mu_i = 1. \tag{1}$$

THEOREM 1. *The Diophantine equation*

$$f(x) = g(x) \tag{2}$$

*has integral solutions, and every solution for which the members of* (2) *do not vanish, is equivalent (in a sense to be defined) to one of the solutions given by*

$$x_{kj} = \alpha_{kj}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}, \qquad k = 1, \cdots, n;\ j = 1, \cdots, m, \tag{3}$$

*where the $\alpha_{kj}$ are arbitrary integers.*

PROOF. Let $x_{kj}=\alpha_{kj}s^{\lambda_k}t^{\mu_k}$. Then by (1), (2) becomes[3] $sf(\alpha)=tg(\alpha)$, which is satisfied identically in the $\alpha_{kj}$ if $s=g(\alpha)$, $t=f(\alpha)$. Hence (3) is a solution of (2).

We now define the concept of equivalent solutions. Suppose $x_{kj}=\rho_{kj}$ is a solution of (2). If there are no integers $b>1$, $\rho'_{kj}$ such that $\rho_{kj}=\rho_{kj}b^{\sigma_k}$, where the $\sigma_k$ are positive integers such that $\sum_{k=1}^{n}\alpha_k\sigma_k$

Received by the editors March 2, 1942.

[1] A. A. Aucoin and W. V. Parker, *Diophantine equations whose members are homogeneous*, this Bulletin, vol. 45 (1939), pp. 330–333.

[2] We need to postulate only the existence of $\lambda_i$ since it may be shown that for $n$ odd $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n-1)/2}d_{i+k}-\lambda_i$ where $d_{n+p}=d_p$, and for $n$ even $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n/2)-1}d_{i+k}-2d'_{n/2+i}-\lambda_i$ where $d'_k=d_k$ for $k\leqq n$, $d'_k=0$ for $k>n$ and $d_{n+p}=d_p$. We must, however, assume that $\mu_i \geqq 0$.

[3] It will be shown later that $s$, $t\neq 0$.

$=\sum_{k=1}^{n}\beta_k\sigma_k$, then $x_{kj}=\rho_{kj}$ is defined to be a primitive solution of (2). If $x_{kj}=\rho_{kj}$ is a primitive solution of (2), then $x_{kj}=\rho_{kj}b^{\sigma_k}$ (derived from the primitive solution), where $b$ is a nonzero integer,[4] $\sigma_k$ are positive integers such that $\sum_{k=1}^{n}\alpha_k\sigma_k=\sum_{k=1}^{n}\beta_k\sigma_k$, is also a solution. Two solutions are said to be equivalent if they may be derived from the same primitive solution.

Suppose now that $x_{kj}=\rho_{kj}$ is a solution of (2). Then $f(\rho)=g(\rho)$. If we choose $\alpha_{kj}=\rho_{kj}$, (3) becomes $x_{kj}=\rho_{kj}[f(\rho)]^{\lambda_k+\mu_k}$ which is equivalent to the given solution $x_{kj}=\rho_{kj}$ provided $f(\rho)\neq 0$, since from (1), $\sum_{k=1}^{n}\alpha_k(\lambda_k+\mu_k)=\sum_{k=1}^{n}\beta_k(\lambda_k+\mu_k)$. If $f(\rho)\neq 0$, then $s,\ t\neq 0$.

As our next topic we suppose that for the functions considered above, $f$ is of degree $-\alpha_p$ and $g$ is of degree $-\beta_p$ in the set $x_{p1}\,x_{p2}\cdots x_{pm}$, where $\alpha_p,\ \beta_p$ are positive integers.[5] Here we let $d_p=\beta_p-\alpha_p$.

THEOREM 2. *The Diophantine equation*

$$f(x) = g(x) \tag{4}$$

*has solutions, and every solution, for which the members of* (4) *do not vanish, is equivalent to a solution given by*

$$x_{kj} = \alpha_{kj}[A(\alpha)B(\alpha)]^{\lambda_k+\mu_k}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}, \tag{5}$$

*where* $A(x)=\prod_{j=1}^{m}x_{pj}^{\alpha_p}$, $B(x)=\prod_{j=1}^{m}x_{pj}^{\beta_p}$, *the* $\alpha_{kj}$ *being arbitrary integers.*

PROOF. If we multiply (4) by $A(x)B(x)$ we have

$$A(x)B(x)f(x) = A(x)B(x)g(x), \tag{6}$$

each member of which is a polynomial. If we let

$$x_{kj} = \alpha_{kj}s^{\lambda_k}t^{\mu_k}, \tag{7}$$

equation (6) becomes[6] $sA(\alpha)B(\alpha)f(\alpha)=tA(\alpha)B(\alpha)g(\alpha)$, which is identically satisfied in the $\alpha_{kj}$ if $s=A(\alpha)B(\alpha)g(\alpha)$, $t=A(\alpha)B(\alpha)f(\alpha)$. Hence (5) is a solution of (4).

It is evident that more than one of the corresponding sets of variables may be of negative degree.

Suppose that $x_{kj}=\rho_{kj}$ is a given solution of (4). Then $f(\rho)=g(\rho)$. If we choose $\alpha_{kj}=\rho_{kj}$, (5) becomes

$$x_{kj}=\rho_{kj}[A(\rho)B(\rho)f(\rho)]^{\lambda_k+\mu_k}$$

which is equivalent to the given solution $x_{kj}=\rho_{kj}$ provided $f(\rho)\neq 0$, since from (1)

[4] If $b=0$, the solution is trivial.

[5] The $\lambda_i$, and hence the $\mu_i$, here are different from those of Theorem 1.

[6] It will be shown later that $s,\ t\neq 0$.

$$\sum_{k=1}^{n} \alpha_k(\lambda_k + \mu_k) - 2\alpha_p(\lambda_p + \mu_p) = \sum_{k=1}^{n} \beta_k(\lambda_k + \mu_k) - 2\beta_p(\lambda_p + \mu_p).$$

If $f(\rho) \neq 0$ then $s,\ t \neq 0$. We now pass on to our next result.

Suppose that $f$ is of degree $-\alpha_p$ in the set $x_{p1}\, x_{p2} \cdots x_{pm}$ and $g$ is of degree $-\beta_q$ in the set $x_{q1}x_{q2} \cdots x_{qm}$ where $\alpha_p$, $\beta_q$ are positive integers. Here[7] $d_p = -(\alpha_p + \beta_p)$, $d_q = \alpha_q + \beta_q$.

THEOREM 3. *The Diophantine equation $f(x) = g(x)$ has solutions, and every solution for which the members do not vanish, is equivalent to a solution given by $x_{kj} = \alpha_{kj}[C(\alpha)D(\alpha)]^{\lambda_k+\mu_k}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}$ where $C(x) = \prod_{j=1}^{m} x_{pj}^{\alpha_p}$, $D(x) = \prod_{j=1}^{m} x_{qj}^{\beta_q}$, the $\alpha_{kj}$ being arbitrary integers.*

The proof is similar to that of Theorem 2.

We can now extend the above methods of solution to nonhomogeneous equations. The nature of this extension is contained in the following theorem.

THEOREM 4. *The Diophantine equation*[8]

$$f(x) = g(x) \tag{8}$$

*where $f(x) = \sum_{h=1}^{m} a_h \prod_{i=1}^{n} x_i^{\alpha_{hi}}$, $g(x) = \sum_{k=1}^{p} b_k \prod_{j=1}^{q} x_j^{\beta_{kj}}$, $a_h$, $b_k$ are integers, $\alpha_{hi}$, $\beta_{kj}$ positive integers, has solutions if there exist positive integers $u_i$, $v_j$, $M$, $N$ such that*

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_{hi}u_i = M + 1, && h = 1, \cdots, m,\\
&\sum_{j=1}^{q} \beta_{kj}u_j = M, && k = 1, \cdots, p,\\
&\sum_{i=1}^{n} \alpha_{hi}v_i = N, && h = 1, \cdots, m,\\
&\sum_{j=1}^{q} \beta_{kj}v_j = N + 1, && k = 1, \cdots, p,
\end{aligned} \tag{9}$$

*and every solution for which the members of (8) do not vanish, is equivalent to a solution given by*

$$x_j = \alpha_j[g(\alpha)]^{u_j}[f(\alpha)]^{v_j}, \tag{10}$$

*where the $\alpha_j$ are arbitrary integers.*

[7] As in the previous theorem the $\lambda_i$ and $\mu_i$ will be different from those of Theorem 1.

[8] A. A. Aucoin and W. V. Parker, op. cit., p. 331. Theorem 4, which generalizes Theorem 2 of the reference may be obtained from that theorem directly by setting $y_j = x_i$.

PROOF. Let $x_j=\alpha_j s^{u_j}t^{v_j}$. Then (8) becomes, by (9), $s^{M+1}t^{N}f(\alpha)=s^{M}t^{N+1}g(\alpha)$, which is satisfied identically in the $\alpha_j$ if $s=g(\alpha)$, $t=f(\alpha)$. Hence (10) is a solution of (8).

When the conditions of (9) are satisfied there also exist non-negative integers $R$, $u_j'$ such that

$$(11)\qquad \sum_{i=1}^{n}\alpha_{hi}u_i' = R,\quad \sum_{j=1}^{q}\beta_{kj}u_j' = R,\quad h=1,\cdots,m;\ k=1,\cdots,p.$$

If $x_j=\rho_j$ is a solution of (8) and there are no integers $d>1$, $u_j'$, $\rho_j'$ such that $\rho_j=\rho_j' d^{u_j'}$ where $u_j'$ satisfies (11), then $x_j=\rho_j$ is defined to be a primitive solution of (8). If $x_j=\rho_j$ is a primitive solution of (8), then $x_j=\rho_j d^{u_j'}$ (derived from the primitive solution) where $d$ $(\neq 0)$ is an integer and $u_j'$ are non-negative integers satisfying (11), is also a solution.

Suppose $x_j=\rho_j$ is a solution of (8). Then $f(\rho)=g(\rho)$. If we choose $\alpha_j=\rho_j$, (10) becomes $x_j=\rho_j[f(\rho)]^{u_j+v_j}$ which is equivalent to the solution $x_j=\rho_j$ since from (9)

$$\sum_{i=1}^{n}\alpha_{hi}(u_i+v_i)=\sum_{j=1}^{q}\beta_{kj}(u_j+v_j)=M+N+1.$$

In Theorems 2 and 3 it is conceivable that the method applies to cases other than those in which an entire set of variables appear in the denominator. The cases are too numerous to consider. It is likewise apparent that the method of Theorem 4 applies when some of the exponents are negative. As there is no typical case the procedure will be illustrated by a particular example.

Consider the equation

$$(12)\qquad ax^3u^5/v^7 + by^4v^4/u^2 = cx^5/w$$

which may be written as

$$(13)\qquad ax^3u^7w + by^4v^{11}w = cx^5u^2v^7.$$

If we let $x=\alpha s^{p_1}t^{q_1}$, $y=\beta s^{p_2}t^{q_2}$, $u=\lambda s^{p_3}t^{q_3}$, $v=\mu s^{p_4}t^{q_4}$, $w=\nu s^{p_5}t^{q_5}$ and substitute in (13) we have

$$s^{3p_1+7p_3+p_5}t^{3q_1+7q_3+q_5}a\alpha^3\lambda^7\nu + s^{4p_2+11p_4+p_5}t^{4q_2+11q_4+q_5}b\beta^4\mu^{11}\nu = s^{5p_1+2p_3+7p_4}t^{5q_1+2q_3+7q_4}c\alpha^5\lambda^2\mu^7.$$

We require an integral solution of the equations

$$3p_1+7p_3+p_5=m,\qquad 3q_1+7q_3+q_5=n+1,$$
$$4p_2+11p_4+p_5=m,\qquad 4q_2+11q_4+q_5=n+1,$$

$$5p_1 + 2p_2 + 7p_4 = m + 1, \qquad 5q_1 + 2q_2 + 7q_4 = n.$$

A solution of these equations is $p_1=4$, $p_2=3$, $p_3=11$, $p_4=7$, $p_5=1$, $q_1=4$, $q_2=3$, $q_3=11$, $q_4=7$, $q_5=3$. Hence a solution of (12) is $x=\alpha s^4t^4$, $y=\beta s^3t^3$, $u=\lambda s^{11}t^{11}$, $v=\mu s^7t^7$, $w=\nu st^3$ where $s=a\alpha^3\lambda^7\nu+b\beta^4\mu^{11}\nu$, $t=c\alpha^5\lambda^2\mu^7$.

If $x=x'$, $y=y'$, $u=u'$, $v=v'$, $w=w'$ is a given solution of (12) and the choice $\alpha=x'$, $\beta=y'$, $\lambda=u'$, $\mu=v'$, $\nu=w'$ is made then $s=t$ and the solution becomes $x=x't^8$, $y=y't^8$, $u=u't^{22}$, $v=v't^{14}$, $w=w't^4$ which is equivalent to the given solution provided $t\neq 0$.

UNITED STATES NAVAL ACADEMY AND
UNIVERSITY OF HOUSTON

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## VECTOR ANALOGUES OF MORERA'S THEOREM

E. F. BECKENBACH

Let the vector

$$\boldsymbol{X} \equiv \boldsymbol{X}(x_1, x_2, x_3) \equiv \boldsymbol{X}(x) \equiv X_1\boldsymbol{i} + X_2\boldsymbol{j} + X_3\boldsymbol{k}$$

be defined and continuous in the domain (non-null connected open set) $D$. Consider the mean-value vector

$$\boldsymbol{X}^{(\rho)}(x) \equiv \frac{1}{|V_\rho|}\int_{V_\rho} \boldsymbol{X}(x+\xi)dV, \tag{1}$$

where $V_\rho$ denotes the sphere

$$\xi_1^2 + \xi_2^2 + \xi_3^2 < \rho^2,$$

and $|V_\rho|$ its volume,

$$|V_\rho| \equiv 4\pi\rho^3/3.$$

The vector (1) can be defined thus for only a part $D_\rho$ of $D$, but this is of no consequence since $\rho$ is arbitrarily small.

Since $\boldsymbol{X}(x)$ is continuous, it follows that $\boldsymbol{X}^{(\rho)}(x)$ has continuous partial derivatives of the first order; these are given by

$$\frac{\partial}{\partial x_p}\boldsymbol{X}^{(\rho)}(x) = \frac{1}{|V_\rho|}\int_{S_\rho} \boldsymbol{X}(x+\rho\alpha)\alpha_p d\sigma, \tag{2}$$

where $S_\rho$ denotes the surface of $V_\rho$ and $\alpha_1$, $\alpha_2$, $\alpha_3$ are the components of the unit vector along the outer normal to $S_\rho$.

Presented to the Society, September 5, 1941 under the title *Vector formulations of Morera's theorem*; received by the editors March 3, 1942.

Mean-value functions are particularly useful in giving sufficiency proofs involving lessening of differentiability conditions. We shall use mean-value vectors to establish two generalizations of Morera's theorem to vector functions.

A vector $\boldsymbol{X}(x)$ having continuous partial derivatives of the first order in the finite domain $D$ has been called[1] a *Newtonian vector* provided it is both irrotational and solenoidal; that is, provided its curl and its divergence both vanish identically in $D$:

$$\text{curl } \boldsymbol{X} \equiv \nabla \times \boldsymbol{X} \equiv \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3}\right)\boldsymbol{i} + \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1}\right)\boldsymbol{j} + \left(\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}\right)\boldsymbol{k} = 0; \tag{3}$$

$$\text{div } \boldsymbol{X} \equiv \nabla \cdot \boldsymbol{X} \equiv \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} = 0. \tag{4}$$

It follows readily[2] that every Newtonian vector is a harmonic vector.

For a vector of components $X_1(x_1, x_2)$, $X_2(x_1, x_2)$, equations analogous to (3) and (4) are the Cauchy-Riemann differential equations

$$\frac{\partial X_2}{\partial x_1} = \frac{\partial X_1}{\partial x_2}, \qquad \frac{\partial X_2}{\partial x_2} = -\frac{\partial X_1}{\partial x_1}$$

for the function

$$f(z) \equiv X_2(x_1, x_2) + iX_1(x_1, x_2), \qquad z \equiv x_1 + ix_2.$$

This fact is taken as justification for considering Newtonian vectors as a generalization of analytic functions of a complex variable.

Let the vector $\boldsymbol{X}(x)$ have continuous partial derivatives of the first order in $D$. By Stokes' theorem, $\boldsymbol{X}(x)$ is irrotational if and only if for each reducible[3] closed curve $C$ in $D$,

$$\int_C \boldsymbol{X} \cdot d\boldsymbol{R} = 0, \tag{5}$$

where $\boldsymbol{R}$ is the vector from the origin to a moving point on $C$. By

[1] D. G. Fulton and G. Y. Rainich, *Generalizations to higher dimensions of the Cauchy integral formula*, American Journal of Mathematics, vol. 54 (1932), pp. 235–241.

[2] See, for instance, H. B. Phillips, *Vector Analysis*, New York, 1933, pp. 149, 153.

[3] A closed curve (or surface) in $D$ is reducible provided it is rectifiable (or of finite area) and can be shrunk continuously in $D$ to a point of $D$. The forms of Stokes' theorem and Green's theorem referred to in this paragraph may be found in H. B. Phillips, loc. cit., pp. 62, 68, 72.

Green's theorem, $\boldsymbol{X}(x)$ is irrotational if and only if for each reducible closed surface $S$ in $D$,

$$\int_S \boldsymbol{n} \times \boldsymbol{X} d\sigma = 0, \tag{6}$$

where $\boldsymbol{n}$ is the unit vector along the outer normal to $S$; and again by Green's theorem, $\boldsymbol{X}(x)$ is solenoidal if and only if for each reducible closed surface $S$ in $D$,

$$\int_S \boldsymbol{n} \cdot \boldsymbol{X} d\sigma = 0. \tag{7}$$

It follows that Newtonian vectors are characterized by (5) and (7), and also by (6) and (7); but (cf. Morera's theorem) *the assumption that $\boldsymbol{X}(x)$ has continuous partial derivatives of the first order now is redundant*,[4] as we shall show. We shall further lessen the sufficiency conditions by replacing the integral conditions (5), (6) and (7) by weaker local conditions which are implied by (5), (6) and (7), respectively.

We shall denote by $S(x, r)$ the sphere with center $(x)$ and radius $r$, and by $C_1(x, r)$, $C_2(x, r)$, $C_3(x, r)$ the circles with center $(x)$ and radius $r$ in planes perpendicular to the $x_1$, $x_2$, $x_3$ axes, respectively. Let

$$\boldsymbol{C}(x, r) \equiv C_1(x, r)\boldsymbol{i} + C_2(x, r)\boldsymbol{j} + C_3(x, r)\boldsymbol{k},$$

and let $o(r^t)$ denote a function (not always the same function) of $r$ such that

$$\lim_{r \to 0} \frac{o(r^t)}{r^t} = 0.$$

THEOREM 1. *Let the vector $\boldsymbol{X}(x)$ be continuous in the finite domain $D$. If*

$$\int_{C(x,r)} \boldsymbol{X} \cdot d\boldsymbol{R} = o(r^2) \tag{8}$$

*and*

$$\int_{S(x,r)} \boldsymbol{n} \cdot \boldsymbol{X} d\sigma = o(r^3) \tag{9}$$

*uniformly in each sphere in $D$, then $\boldsymbol{X}(x)$ is a Newtonian vector.*

[4] See H. B. Phillips, loc. cit., pp. 177, 178, for a proof of the result that if the continuous vector function $\boldsymbol{X}(x)$ satisfies (5) and (7), then $\boldsymbol{X}(x)$ is a Newtonian vector.

PROOF. Since (8) and (9) hold uniformly in each sphere in $D$, we have

$$\frac{1}{|V_\rho|}\int_{V_\rho}\left(\int_{C_{(x,r)}} \boldsymbol{X}\cdot d\boldsymbol{R}\right)dV = o(r^2)$$

and

$$\frac{1}{|V_\rho|}\int_{V_\rho}\left(\int_{S(x,r)} \boldsymbol{n}\cdot\boldsymbol{X}d\sigma\right)dV = o(r^3)$$

in $D_\rho$. Interchanging the order of integration, we obtain

$$\int_{C_{(x,r)}} \boldsymbol{X}^{(\rho)}\cdot d\boldsymbol{R} = o(r^2) \tag{10}$$

and

$$\int_{S(x,r)} \boldsymbol{n}\cdot\boldsymbol{X}^{(\rho)}d\sigma = o(r^3). \tag{11}$$

Since $\boldsymbol{X}^{(\rho)}(x)$ has continuous partial derivatives of the first order in $D_\rho$, about any point $(x^{(0)})$ in $D_\rho$ we have the finite Taylor development

$$\boldsymbol{X}^{(\rho)}(x) = \boldsymbol{X}^{(\rho)}(x^{(0)}) + \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_1}(x_1 - x_1^{(0)}) + \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_2}(x_2 - x_2^{(0)}) + \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_3}(x_3 - x_3^{(0)}) + o(r),$$

where the partial derivatives are evaluated at $(x^{(0)})$. Computations yield

$$\int_{C_{(x^{(0)},r)}} \boldsymbol{X}^{(\rho)}\cdot d\boldsymbol{R} = \pi r^2 \operatorname{curl} \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^2) \tag{12}$$

and

$$\int_{S(x^{(0)},r)} \boldsymbol{n}\cdot\boldsymbol{X}^{(\rho)}d\sigma = (4/3)\pi r^3 \operatorname{div} \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^3). \tag{13}$$

Now (10)–(13) give

$$\operatorname{curl} \boldsymbol{X}^{(\rho)}(x) = 0$$

and

$$\operatorname{div} \boldsymbol{X}^{(\rho)}(x) = 0$$

throughout $D_\rho$; that is, $\boldsymbol{X}^{(\rho)}(x)$ is a Newtonian vector. Hence $\boldsymbol{X}(x)$, the uniform limit of $\boldsymbol{X}^{(\rho)}(x)$ as $\rho \to 0$, is a Newtonian vector.

THEOREM 2. *Let the vector $\boldsymbol{X}(x)$ be continuous in the finite domain $D$. If*

$$\int_{S(z,r)} n \times \boldsymbol{X} d\sigma = o(r^3)$$

*and*

$$\int_{S(z,r)} n \cdot \boldsymbol{X} d\sigma = o(r^3)$$

*uniformly in each sphere in $D$, then $\boldsymbol{X}(x)$ is a Newtonian vector.*

PROOF. The proof is analogous to that of Theorem 1, with

$$\int_{S(x^{(0)},r)} n \times \boldsymbol{X}^{(\rho)} d\sigma = (4/3)\pi r^3 \operatorname{curl} \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^3)$$

in place of (12).

THE UNIVERSITY OF MICHIGAN

# ON THE INTERIORITY OF REAL FUNCTIONS[1]

G. T. WHYBURN

If $X$ and $Y$ are metric spaces, a continuous transformation $f(X) = Y$ is said to be interior at a point $x$ of $X$ provided that if $U$ is any open subset of $X$ containing $x$, $f(x)$ is interior to $f(U)$ in $Y$. In 1928 Kuratowski[2] proved a theorem essentially to the effect that if $X$ is compact, the set $G$ of all points $y$ in $Y$ such that $f$ is interior at all points of $f^{-1}(y)$ is a $G_\delta$-set dense in $Y$. We first establish the following related result.

THEOREM. *If $f(x)$ is a real-valued continuous function defined on a locally connected separable metric space $M$ (therefore transforming $M$ into a set $Y$ of real numbers), there exists a countable subset $C$ of $Y$ such that $f$ is interior at every point of $M - f^{-1}(C)$.*

PROOF. For each $y \in Y$ let $Y_1$ and $Y_2$ be the sets of all numbers in $Y$ which are less than $y$ and greater than $y$, respectively. Let $M_1(y) = f^{-1}(Y_1)$, $M_2(y) = f^{-1}(Y_2)$. Now there must exist[3] a countable subset $C$ of $Y$ such that if $y$ is any point of $Y - C$,

$$f^{-1}(y) \subset \overline{M_1(y)} \cdot \overline{M_2(y)}.$$

For if not, there would exist an uncountable subset $K$ of $Y$ such that for any $y \in K$ there is a point $p_y \in f^{-1}(y)$ which fails to be a limit point either of $M_1(y)$ or of $M_2(y)$. Clearly we may suppose that for an uncountable subset of $K_1$ of $K$, $p_y \cdot \overline{M_1(y)} = 0$. Now since $M$ is separable and metric, there exists a countable sequence $R_1, R_2, \cdots$ of open sets in $M$ such that if $U$ is any open set in $M$ and $p \in U$, there is an $m$ such that $p \in R_m \subset U$. Hence, for each $y \in K_1$ there exists an integer $m_y$ such that $p_y \subset R_{m_y}$ and $R_{m_y} \cdot M_1(y) = 0$. But if $y$, $y' \in K_1$, $y' < y$, we have $p_{y'} \subset M_1(y)$ so that $m_y \neq m_{y'}$. This is absurd, since $K_1$ was uncountable. Hence we have established the existence of the set $C$ as asserted above.

We shall show that $f$ is interior at any point $x$ of $M - f^{-1}(C)$. To this end let $U$ be any open set in $M$ containing $x$ and let $y = f(x)$. Since $M$ is locally connected, there exists a connected open subset $V$

Presented to the Society, September 8, 1939; received by the editors March 4, 1942.

[1] This paper, in slightly different form, was sent to the editors of Fundamenta Mathematicae in June, 1939.

[2] See Fundamentae Mathematicae, vol. 11 (1926), p. 176.

[3] This could be established also with the aid of a lemma of Zarankiewicz. See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

of $M$ with $x \subset V \subset U$. Now $V$ contains points $x_1$ and $x_2$ of $M_1(y)$ and $M_2(y)$, respectively. Since $f(V)$ is connected and contains $f(x_1)$ and $f(x_2)$, it therefore contains the interval $[f(x_1), f(x_2)]$. Hence $f(x)$, which is interior to this interval, is interior to $f(V)$ and hence surely to $f(U)$.

Examples are easily constructed to show that our conclusion is invalid in the absence of either the local connectedness condition on $M$ or the linearity restriction on the image space.

However, we can extend the theorem considerably. First we note the following corollary. (Note: All sets referred to in this paper are assumed separable and metric.)

COROLLARY. *Let $f(M) = Y = \sum_{i=1}^{\infty} L_i$ be continuous where $M$ is locally connected, each $L_i$ is open in $Y$ and is homeomorphic with a subset of a line and $L_i \cdot L_j = 0$ for $i \neq j$. There exists a countable subset $C$ of $L$ such that $f$ is interior at every point of $M - f^{-1}(C)$.*

For under the conditions on the $L_i$ it results at once that $Y$ is homeomorphic with a subset of a line and hence with a set of real numbers. (For if $L_i'$ is a subset of the open interval $(i-1, i)$ homeomorphic with $L_i$, $i = 1, 2, \cdots$, clearly $Y$ is homeomorphic with $\sum L_i'$.) Thus $f$ is effectively a real continuous function on $M$ and the corollary follows from the preceding theorem.

THEOREM. *Let $f(M) = K$ be continuous where $M$ is locally connected and let $D$ be the closure of the set of all points of order greater than 2 of $K$. There exists a countable subset $C$ of $K - D$ such that $f$ is interior at every point of $M - f^{-1}(D + C)$.*

PROOF. By continuity of $f$, $M - f^{-1}(D) = M'$ is open in $M$ and hence is locally connected. Consider the mapping $f(M') = K - D = K'$. Since every point of $K'$ is of order at most 2, $K'$ is locally connected. Hence the components $R_1, R_2, \cdots$ of $K'$ are open (thus countable). Further, each $R_i$ is homeomorphic with a subset of a line, since it is a connected set all of whose points are of order at most 2. Thus by the above corollary there exists a countable subset $C$ of $K'$ such that $f$ is interior at all points of $M' - f^{-1}(C)$. Obviously this yields our theorem.

COROLLARY. *If $D$ is countable, there exists a countable subset $C$ of $K$ such that $f$ is interior at all points of $M - f^{-1}(C)$.*

COROLLARY. *Any continuous mapping $f(M) = K$ of a locally connected set $M$ into a graph $K$ is interior at all points of $M - f^{-1}(C)$, where $C$ is some countable subset of $K$.*

If $X$ and $Y$ are topological spaces, $X_1$ is a subset of $X$ and $f(X_1) \subset Y$ and $\phi(X) \subset Y$ are single-valued continuous transformations, $\phi(x)$ is said to be an extension of $f$ to $X$ provided $f(X) = \phi(x)$ for $x \in X_1$. Extension theorems in various forms for continuous transformations are well known.[4] However, not much consideration seems to have been given to the important problem of carrying over to an extension $\phi$ of $f$ properties which $f$ may enjoy in addition to continuity. In this note a beginning in this direction is made with the following theorem.

THEOREM. *Let $M$ be a subcontinuum of a cyclic locally connected compact continuum $L$ and let $f(M) = (0,1)$ be continuous and such that for any $y$, $0 \leqq y \leqq 1$, $L - f^{-1}(y)$ is connected. There exists an extension $\phi(L) = (0, 1)$ of $f$ to $L$ which is interior at every point of $L - M$ and at every point of $M$ where $f$ is interior.*[5] *(Thus if $f$ is interior on $M$, $\phi$ is interior on $L$.)*

PROOF. Let us decompose $L$ upper semi-continuously[6] into the subsets $[f^{-1}(y)]$, $0 \leqq y \leqq 1$, of $M$ and individual points of $L - M$. Call $L'$ the hyperspace of this decomposition and let $h(L) = L'$ be the associated transformation. Clearly $h(M)$ is a simple arc $a'b'$ joining $a' = hf^{-1}(0)$ and $b' = hf^{-1}(1)$. By a theorem of the author,[7] there exists a non-alternating interior retracting transformation $g(x)$ retracting the cyclic chain $C(a,'b')$ in $L'$ into $a'b'$. But since no one of the sets $f^{-1}(y)$ disconnects $L$ and $L$ is cyclic, it follows that $L'$ is cyclic and hence $L' = C(a', b')$. Thus $g(x)$ retracts $L'$ into $a'b'$.

Let us define

$$\phi(x) = fh^{-1}gh(x), \qquad x \in L.$$

Then $\phi$ has the required properties. For if $x \in M$, we have $h(x) \in a'b'$ so that $gh(x) = h(x)$; whence $h^{-1}gh(x) = h^{-1}h(x) = f^{-1}f(x)$ so that $fh^{-1}gh(x) = f(x)$. This proves $\phi(x) \equiv f(x)$ on $M$. To show that $\phi$ is interior at any point $x$ where $f$ is interior or at any point of $L - M$, let $x \in U$ where $U$ is open in $L$. If $x \in M$, then since $f$ is interior at $x$, $h[U \cdot M]$ contains an open subset $V$ of $a'b'$ about $h(x)$. Hence $g(V) = V$

[4] See, for example, F. Hausdorff, Fundamenta Mathematicae, vol. 30 (1938), p. 40, and C. Kuratowski, ibid., p. 48.

[5] It is supposed that $L$ is imbedded in a metric space and (0, 1) denotes the interval $0 \leqq y \leqq 1$ of the real numbers. A continuum is *cyclic* if it has no cut point.

[6] See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, chap. 5; note also references to P. Alexandroff and to C. Kuratowski.

[7] See Whyburn, *Non-alternating interior retracting transformations*, Annals of Mathematics, (2), vol. 40 (1939), pp. 914–921.

is open in $a'b'$ and $h^{-1}g(V) \supset x$ and is open in $M$. Since $f$ is interior at $x$, $fh^{-1}g(V)$ has $f(x)$ as an interior point; and since this set surely is contained in $fh^{-1}gh(U) = \phi(U)$, it follows that $\phi(x)$ is interior to $\phi(U)$. If $x \in L - M$, we may suppose $U \subset L - M$. Then $h$ is topological on $U$, so that $h(U)$ is open in $L'$. Thus $gh(U)$ is open in $a'b'$, since $g$ is interior on $L'$. Hence $h^{-1}gh(U)$ is open in $M$ and $fh^{-1}gh(U) = \phi(U)$ has $\phi(x)$ as an interior point, since $f$ is interior at $x$.

It is clear that essentially the same argument suffices to establish the following somewhat more general extension theorem:

*Let $M$ be a subcontinuum of a locally connected continuum $L$ of the form $L = C(a, b)$ where $a, b \in M$. Let $f(M) = (0, 1)$ be such that $f(a) = 0$, $f(b) = 1$ and for any $y$ with $0 < y < 1$, every component of $L - f^{-1}(y)$ contains either $a$ or $b$. There exists an extension $\phi(L) = (0, 1)$ of $f$ to $L$ which is interior at every point of $L - M$ and also at every point of $M$ where $f$ is interior.*

UNIVERSITY OF VIRGINIA

# THE MODULAR SPACE DETERMINED BY A POSITIVE FUNCTION

R. W. BARNARD AND H. H. GOLDSTINE

At the suggestion of T. H. Hildebrandt the authors undertook to determine the nature of the space of modular functions of E. H. Moore when the range $\mathfrak{P}$ is taken to be the infinite interval $-\infty < x < +\infty$ and the base matrix $\epsilon$ to be of the form

$$\epsilon(x, y) = \int_{-\infty}^{+\infty} e^{i(x-y)t} dV(t), \tag{1}$$

where $V$ is a monotonically increasing bounded function. This form of $\epsilon$ is suggested by the work of Bochner on positive functions.[1] In this note we determine the form of functions modular as to $\epsilon$ and of the $J$-integral.

To avoid, at first, convergence questions we turn our attention to functions $\phi$ finite as to $\epsilon$, that is, functions of the form

$$\phi(x) = \sum_{j=1}^{n} \epsilon(x, y_j)a_j = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV(t), \tag{2}$$

where

$$\lambda(t) = \sum_{j=1}^{n} a_j e^{-iy_j t}. \tag{3}$$

In the formulas (2) and (3) the $a_j$ are arbitrary constants and the $y_j$ are points on the interval $(-\infty, +\infty)$. It is known from standard results in the theory of modular and finite functions[2] that every function $\phi$ finite as to $\epsilon$ is modular and that

$$\begin{aligned} N\phi = J\bar{\phi}\phi &= \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k)a_k, \\ J\bar{\phi}_1\phi_2 &= [N(\phi_1 + \phi_2) - N(\phi_1 - \phi_2) - iN(\phi_1 + i\phi_2) \\ &\qquad + iN(\phi_1 - i\phi_2)]/4. \end{aligned} \tag{4}$$

Calculating the values of $N\phi$ and $J\bar{\phi}_1\phi_2$, we see that

$$J\bar{\phi}_1\phi_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV, \qquad N\phi = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

Received by the editors March 7, 1942.

[1] S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen, vol. 108 (1933), pp. 378–410.

[2] E. H. Moore, *General Analysis*, Part II, Philadelphia, 1939, pp. 94 ff.

To determine the form of an arbitrary modular function $\mu$ we consider a sequence

$$\phi_n(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda_n dV$$

of functions finite as to $\epsilon$, converging mode 2 to a modular function[3] $\mu$. Since $\phi_n$ converges strongly, it follows that

$$\lim_{m,n} N(\phi_m - \phi_n) = \lim_{m,n} \int_{-\infty}^{+\infty} |\lambda_m - \lambda_n|^2 dV = 0,$$

and hence there exists a measurable function $\lambda$ such that $\lambda^2$ is integrable with respect to $V$ and[4]

$$\lim_{n} \int_{-\infty}^{+\infty} |\lambda_n - \lambda|^2 dV = 0.$$

With the help of Schwarz' inequality one sees that

$$\mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV, \qquad N\mu = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

THEOREM 1. *To each modular function $\mu$ there corresponds a measurable function $\lambda$ such that $\lambda^2$ is integrable with respect to $V$ and*

$$(5) \qquad \mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV(t) \quad \textit{and} \quad N\mu = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

*Conversely, if $\lambda$ is measurable and $\lambda^2$ integrable with respect to $V$, then the first of the formulas (5) defines a modular function $\mu$ for which the second of these formulas is valid. If $\mu_1$, $\mu_2$ are two modular functions, then*

$$(6) \qquad J\bar{\mu}_1\mu_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV$$

*where $\lambda_1$, $\lambda_2$ are the square integrable functions associated with $\mu_1$, $\mu_2$.*

It remains to prove only the latter part of the theorem. To do this let $\xi(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda dV$, where $\lambda$ is any measurable function such that $\lambda^2$ is integrable with respect to $V$, and let $x_j$, $a_j$ $(j = 1, 2, \cdots, n)$ be constants such that

[3] Ibid., p. 116.

[4] E. W. Hobson, *The Theory of Functions of a Real Variable*, 2d edition, 1926, vol. II, p. 246.

$$\sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV \leqq 1.$$

It then follows with the help of Schwarz' inequality that

$$\left| \sum_{j=1}^{n} \bar{a}_j \xi(x_j) \right|^2 = \left| \int_{-\infty}^{+\infty} \left( \sum_{j=1}^{n} \bar{a}_j e^{ix_j t} \right) \lambda dV \right|^2 \leqq \int_{-\infty}^{+\infty} |\lambda|^2 dV,$$

and hence $\xi$ is modular.[5] The formula (6) follows at once from the second equation (5) and equation (4).

Finally, we seek conditions that the matrix $\epsilon$ should be proper. These are contained in the following result:

THEOREM 2. *The base matrix $\epsilon$ is proper if the measure function $V$ is such that every set $E$ whose complement has zero measure has a finite limit point.*

It is clear that

$$0 = \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV$$

implies the vanishing of the analytic function

$$\sum_{j=1}^{n} a_j e^{-ix_j t} \tag{7}$$

for almost all $t$. If the constants $a_j$ were not all zero, the expression (7) would have a non-finite number of zeros in a bounded interval, which is false, and hence $a_1, \cdots, a_n$ are all zero and $\epsilon$ is a proper matrix.

THE UNIVERSITY OF CHICAGO AND
THE UNIVERSITY OF MICHIGAN

[5] Ibid., p. 84.

## A CORRECTION TO "ON THE ITERATION OF LINEAR HOMOGENEOUS TRANSFORMATIONS"

ARNOLD DRESDEN

The theorem proved in the note entitled *On the iteration of linear homogeneous transformations*, published in this Bulletin, vol. 48 (1942), pp. 577–579, was proved by Rufus Oldenburger, in a paper on *Infinite powers of matrices and characteristic roots*, which appeared in the Duke Mathematical Journal, vol. 6 (1940), pp. 357–361. My apologies are due to Professor Oldenburger.

SWARTHMORE COLLEGE

Received by the editors August 30, 1942.

## AN ADDITION TO "GENERALIZATIONS OF THE BERNOULLI POLYNOMIALS AND NUMBERS AND CORRESPONDING SUMMATION FORMULAS"

TOMLINSON FORT

In the third paragraph of my paper *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas*, published in this Bulletin, vol. 48 (1942), pp. 567–574, no explicit statement is made of the fact that it is assumed that $P$ and $Q$ as operators on $x$ obey the laws: $PF(x+c)\,]_{c=0} = PF(x)$; $QF(x+c)\,]_{c=0} = QF(x)$.

Also the statement $f_n(0)=0$, should read $f_n(0)=0$, $n>0$.

LEHIGH UNIVERSITY

Received by the editors November 4, 1942.

## ERRATA, VOLUME 48

M. R. Hestenes, *The problem of Bolza in the calculus of variations*, pp. 57–75.

p. 63, first display line. Instead of $g(t)$ read $g(a)$.

p. 72, line 32. Instead of by $E_0^*$ the set $\cdots$ read by $\mathcal{E}_0^*$ the set $\cdots$.

Isaac Opatowski, *Confluent hypergeometric functions and Markoff chains*, abstract 48-5-179.

p. 365, line 2. Instead of $k_i=0$ read $k_0=0$.

A. N. Lowan, N. Davids and A. Levenson, *Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula*, pp. 739–743.

p. 742, last display line. Instead of $x_k^3$, in the denominator, read $x_k+3$.

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